

On finding an obstacle with the Leontovich boundary condition via the time domain enclosure method

Masaru IKEHATA*

November 17, 2015

Abstract

An inverse obstacle scattering problem for the wave governed by the Maxwell system in the time domain, in particular, over a finite time interval is considered. It is assumed that the electric field \mathbf{E} and magnetic field \mathbf{H} which are solutions of the Maxwell system are generated only by a current density at the initial time located not far a way from an unknown obstacle. The obstacle is embedded in a medium like air which has constant electric permittivity ϵ and magnetic permeability μ . It is assumed that the fields on the surface of the obstacle satisfy the impedance-or the Leontovich boundary condition $\boldsymbol{\nu} \times \mathbf{H} - \lambda \boldsymbol{\nu} \times (\mathbf{E} \times \boldsymbol{\nu}) = \mathbf{0}$ with λ an unknown positive function and $\boldsymbol{\nu}$ the unit outward normal. The observation data are given by the electric field observed at the same place as the support of the current density over a finite time interval. It is shown that an indicator function computed from the electric fields corresponding two current densities enables us to know: the distance of the center of the common spherical support of the current densities to the obstacle; whether the value of the impedance λ is greater or less than the special value $\sqrt{\epsilon/\mu}$.

AMS: 35R30, 35L50, 35Q61, 78A46, 78M35

KEY WORDS: enclosure method, inverse obstacle scattering problem, electromagnetic wave, obstacle, Maxwell's equations, Leontovich boundary condition

1 Introduction

In this paper, we consider an inverse obstacle scattering problem for the wave governed by the Maxwell system in the time domain, in particular, over a finite time interval. The formulation of the problem basically follows that of [9].

We assume that the electric field \mathbf{E} and magnetic field \mathbf{H} are generated only by the current density \mathbf{J} at the initial time located not far a way from an unknown obstacle. The obstacle is embedded in a medium like air which has constant electric permittivity $\epsilon(>0)$ and magnetic permeability $\mu(>0)$. On the surface of the obstacle unlike [9] it is assumed that the fields \mathbf{E} and \mathbf{H} satisfy the impedance- or the Leontovich boundary condition ([4, 12]).

*Laboratory of Mathematics, Institute of Engineering, Hiroshima University, Higashi-Hiroshima 739-8527, JAPAN

Let us formulate the problem more precisely. We denote by D the unknown obstacle. We assume that D is a non empty bounded open set of \mathbf{R}^3 with C^2 -boundary such that $\mathbf{R}^3 \setminus \overline{D}$ is connected.

Let $0 < T < \infty$. The governing equations of \mathbf{E} and \mathbf{H} over the time interval $]0, T[$ take the form

$$\left\{ \begin{array}{ll} \epsilon \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} = \mathbf{J} & \text{in } (\mathbf{R}^3 \setminus \overline{D}) \times]0, T[, \\ \mu \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0} & \text{in } (\mathbf{R}^3 \setminus \overline{D}) \times]0, T[, \\ \mathbf{E}|_{t=0} = \mathbf{0} & \text{in } \mathbf{R}^3 \setminus \overline{D}, \\ \mathbf{H}|_{t=0} = \mathbf{0} & \text{in } \mathbf{R}^3 \setminus \overline{D} \end{array} \right. \quad (1.1)$$

and

$$\boldsymbol{\nu} \times \mathbf{H} - \lambda \boldsymbol{\nu} \times (\mathbf{E} \times \boldsymbol{\nu}) = \mathbf{0} \quad \text{on } \partial D \times]0, T[. \quad (1.2)$$

Note that $\boldsymbol{\nu}$ denotes the unit outward normal to ∂D . To ensure the solvability of the initial boundary value problem (1.1) and (1.2) by the theory of C_0 contraction semigroup [13] we assume that $\lambda \in C^1(\partial D)$ and satisfies $\lambda(x) \geq 0$ for all $x \in \partial D$. Under the condition $\mathbf{J} \in C^1([0, T], L^2(\mathbf{R}^3 \setminus \overline{D})^3)$ we have the unique solution (\mathbf{E}, \mathbf{H}) which belongs to $C^1([0, T], L^2(\mathbf{R}^3 \setminus \overline{D})^3 \times L^2(\mathbf{R}^3 \setminus \overline{D})^3)$ with $(\nabla \times \mathbf{E}(t), \nabla \times \mathbf{H}(t)) \in L^2(\mathbf{R}^3 \setminus \overline{D})^3 \times L^2(\mathbf{R}^3 \setminus \overline{D})^3$ and (1.2) is satisfied in the sense of the trace [5].

The boundary condition in (1.2) is called the Leontovich boundary condition and equivalent to the condition

$$\boldsymbol{\nu} \times (\mathbf{H} \times \boldsymbol{\nu}) + \lambda \boldsymbol{\nu} \times \mathbf{E} = \mathbf{0} \quad \text{on } \partial D \times]0, T[.$$

In what follows we use these equivalent forms without mentioning explicitly.

The mathematical role of the existence of the impedance λ can be seen by formally differentiating the energy

$$\mathcal{E}(t) = \frac{1}{2} \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2) dx.$$

By the help of the divergence theorem and (1.2), we have

$$\mathcal{E}'(t) = - \int_{\partial D} \lambda |\boldsymbol{\nu} \times (\mathbf{E} \times \boldsymbol{\nu})|^2 dS + \int_{\mathbf{R}^3 \setminus \overline{D}} \mathbf{J} \cdot \mathbf{E} dx.$$

Thus if $\mathbf{J}(t) = 0$ for $t \in [\delta, T]$ with a small $\delta > 0$, we have $\mathcal{E}'(t) \leq 0$ therein. The solution loses its energy on the surface of the obstacle.

There should be several choices of current density \mathbf{J} as a model of antenna [1, 2]. In this paper, as considered in [9] we assume that \mathbf{J} takes the form

$$\mathbf{J}(x, t) = f(t) \chi_B(x) \mathbf{a}, \quad (1.3)$$

where \mathbf{a} is an arbitrary unit vector; B is a (very small) open ball satisfying $\overline{B} \cap \overline{D} = \emptyset$ and χ_B denotes the characteristic function of B ; $f \in C^1[0, T]$ with $f(0) = 0$.

We consider the following problem.

Problem. Fix a large (to be determined later) $T < \infty$. Generate the solutions \mathbf{E} and \mathbf{H} of system (1.1) and (1.2) by the source \mathbf{J} having the form (1.3) with $f \neq 0$ and observe \mathbf{E} on B over the time interval $]0, T[$. The unit vector \mathbf{a} on (1.3) should be taken from a set of three linearly independent vectors. Extract information about the geometry of D and the qualitative state of the distribution of λ over ∂D from the observed data.

As the author knows there is no result for this problem. Note that in [9] the perfect conductive boundary condition $\boldsymbol{\nu} \times \mathbf{E} = \mathbf{0}$ on ∂D which is the extreme case $\lambda = +\infty$ of the Leontovich boundary condition has been considered. Thus, therein extracting the geometry of D is the main interest. Here we wish to know the *qualitative state* of the surface of the obstacle which is described by the unknown function λ on ∂D .

1.1 The statement of the results

Let \mathbf{E}_0 and \mathbf{H}_0 be the solutions of

$$\left\{ \begin{array}{ll} \epsilon \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} = \mathbf{J} & \text{in } \mathbf{R}^3 \times]0, T[, \\ \mu \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0} & \text{in } \mathbf{R}^3 \times]0, T[, \\ \mathbf{E}|_{t=0} = \mathbf{0} & \text{in } \mathbf{R}^3, \\ \mathbf{H}|_{t=0} = \mathbf{0} & \text{in } \mathbf{R}^3. \end{array} \right. \quad (1.4)$$

Using \mathbf{E} and \mathbf{E}_0 on B over time interval $]0, T[$, we introduce the indicator function of the enclosure method in this paper

$$I_{\mathbf{f}}(\tau, T) = \int_B \mathbf{f}(x, \tau) \cdot (\mathbf{W}_e - \mathbf{V}_e) dx, \quad (1.5)$$

where

$$\mathbf{f}(x, \tau) = -\frac{\tau}{\epsilon} \int_0^T e^{-\tau t} \mathbf{J}(x, t) dt, \quad (1.6)$$

$$\mathbf{W}_e = \mathbf{W}_e(x, \tau) = \int_0^T e^{-\tau t} \mathbf{E}(x, t) dt \quad (1.7)$$

and

$$\mathbf{V}_e = \mathbf{V}_e(x, \tau) = \int_0^T e^{-\tau t} \mathbf{E}_0(x, t) dt. \quad (1.8)$$

Note that we have

$$\mathbf{f}(x, \tau) = -\frac{\tau}{\epsilon} \tilde{f}(\tau) \chi_B(x) \mathbf{a}, \quad (1.9)$$

where

$$\tilde{f}(\tau) = \int_0^T e^{-\tau t} f(t) dt. \quad (1.10)$$

Before describing the main result we introduce two conditions (A.I) and (A.II) listed below:

$$(A.I) \exists C > 0 \quad \lambda(x) \geq \sqrt{\frac{\epsilon}{\mu}} + C \text{ for all } x \in \partial D;$$

$$(A.II) \exists C > 0 \exists C' > 0 \quad C' \leq \lambda(x) \leq \sqrt{\frac{\epsilon}{\mu}} + C \text{ for all } x \in \partial D.$$

Define $\text{dist}(D, B) = \inf_{x \in D, y \in B} |x - y|$.

Theorem 1.1. *Let \mathbf{a}_j , $j = 1, 2$ be two linearly independent unit vectors. Let $\mathbf{J}_j(x, t) = f(t)\chi_B(x)\mathbf{a}_j$ and $f \in C^1[0, T]$ with $f(0) = 0$ satisfy*

$$\exists \gamma \in \mathbf{R} \quad \liminf_{\tau \rightarrow \infty} \tau^\gamma |f(\tau)| > 0. \quad (1.11)$$

Let \mathbf{f}_j , $j = 1, 2$ denote the \mathbf{f} given by (1.9) and (1.10) with f above and $\mathbf{a} = \mathbf{a}_j$.

Then, we have:

(i) For all $T \leq 2\sqrt{\mu\epsilon}\text{dist}(D, B)$ it holds that

$$\lim_{\tau \rightarrow \infty} e^{\tau T} \sum_{j=1}^2 I_{\mathbf{f}_j}(\tau, T) = 0;$$

(ii) if λ satisfies (A.I), then for all $T > 2\sqrt{\mu\epsilon}\text{dist}(D, B)$ it holds that

$$\lim_{\tau \rightarrow \infty} e^{\tau T} \sum_{j=1}^2 I_{\mathbf{f}_j}(\tau, T) = \infty;$$

(iii) if λ satisfies (A.II), then for all $T > 2\sqrt{\mu\epsilon}\text{dist}(D, B)$ it holds that

$$\lim_{\tau \rightarrow \infty} e^{\tau T} \sum_{j=1}^2 I_{\mathbf{f}_j}(\tau, T) = -\infty.$$

Moreover, in case of both (ii) and (iii) we have, for all $T > 2\sqrt{\mu\epsilon}\text{dist}(D, B)$

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log \left| \sum_{j=1}^2 I_{\mathbf{f}_j}(\tau, T) \right| = -2\sqrt{\mu\epsilon}\text{dist}(D, B). \quad (1.12)$$

Theorem 1.1 says that the T in the problem should be all T satisfying

$$T > 2\sqrt{\mu\epsilon}\text{dist}(D, B).$$

In particular, if we have a known upper bound M such that $M > \text{dist}(D, B)$, then we can choose $T = 2\sqrt{\mu\epsilon}M$ and know the exact value of $\text{dist}(D, B)$ via formula (1.12).

Moreover, we have another characterization of $\text{dist}(D, B)$. Let M be the same positive constant as above. Assume that λ satisfies (A.I) or (A.II). Given $f \in C^1[0, 2\sqrt{\mu\epsilon}M]$ with $f(0) = 0$ define

$$\tilde{f}_T(\tau) = \int_0^T e^{-\tau t} f(t) dt, \quad 0 < T < 2\sqrt{\mu\epsilon}M.$$

We assume that (1.11) is satisfied with $\tilde{f} = \tilde{f}_{2\sqrt{\mu\epsilon}M}$. Generate \mathbf{E} and \mathbf{H} over time interval $]0, 2\sqrt{\mu\epsilon}M[$ using $\mathbf{J}(x, t) = f(t)\chi_B(x)\mathbf{a}_j$. Measure \mathbf{E} on B over time interval $]0, 2\sqrt{\mu\epsilon}M[$. Compute also \mathbf{E}_0 on B for \mathbf{J} above over the same time interval. Using those fields, for each $T \in]0, 2\sqrt{\mu\epsilon}M[$ compute \mathbf{W}_e and \mathbf{V}_e given by (1.7) and (1.8), respectively. Denote by \mathbf{f}_j^T the \mathbf{f} given by (1.6) and compute $I_{\mathbf{f}_j^T}(\tau, T)$ given by (1.5) for $\mathbf{f} = \mathbf{f}_j^T$.

Then, it is easy to see that (1.11) is satisfied also with $\tilde{f} = \tilde{f}_T$ for each $T \in]0, 2\sqrt{\mu\epsilon}M[$. Therefore (i)-(iii) in Theorem 1.1 yield the formula

$$]0, 2\sqrt{\mu\epsilon} \text{dist}(D, B)] = \left\{ T \in]0, 2\sqrt{\mu\epsilon}M[\left| \lim_{\tau \rightarrow \infty} e^{\tau T} \sum_{j=1}^2 I_{\mathbf{f}_j^T}(\tau, T) = 0 \right. \right\}. \quad (1.13)$$

Formula (1.13) has a similarity in the style as that of the *original* enclosure method applied to, for example, the Laplace equation, see (1.3) in [6]. See also [8] for similar statements to (ii) and (iii) in Theorem 1.1 for scalar wave equations in the whole space. Note that $\text{dist}(D, B) = d_{\partial D}(p) - \eta$ where p and η are the center and radius of B , respectively and $d_{\partial D}(p) = \inf_{x \in \partial D} |x - p|$. Thus knowing $\text{dist}(D, B)$ is equivalent to knowing $d_{\partial D}(p)$.

We think that (i), (ii) and (iii) implicitly represent the *finite propagation property* of the wave governed by the Maxwell system. It means that if $T \leq 2\sqrt{\mu\epsilon} \text{dist}(D, B)$, then one can not get a qualitative information about the state of the surface of the obstacle (by using the enclosure method). This is consistent with that the propagation speed of the Maxwell system is given by $1/\sqrt{\mu\epsilon}$. However, note that in the proof presented in this paper we do not make use of the finite propagation property.

The results as stated in (ii) and (iii) together with formula (1.12) can be considered as an extension of the corresponding results in [7]. More precisely, therein we considered an inverse obstacle scattering problem for the wave governed by the classical wave equation $\partial_t^2 u - \Delta u = 0$ outside an obstacle which we denote by D again. The wave u as a solution of the equation satisfies $\partial u / \partial \boldsymbol{\nu} - \gamma \partial_t u - \beta u = 0$ on $\partial D \times]0, T[$. It is assumed that γ and β are essentially bounded functions on ∂D and $\gamma \geq 0$. Using an indicator function which can be computed from a wave generated by a single set of initial data and a special solution of the modified Helmholtz equation, we found that $\gamma \equiv 1$ is the special value as same as $\lambda \equiv \sqrt{\epsilon/\mu}$ of (A.I) and (A.II). It means that the indicator function therein changes the behaviour according to whether γ is greater or less than 1.

For the computation of indicator function (1.5) we need \mathbf{E}_0 on B over time interval $]0, T[$. This can be done by solving explicitly and analytically the initial value problem (1.4).

Here we present another way which introduces an indicator function using an approximation of (1.8).

Let \mathbf{V}_e^0 be the weak solution of

$$\frac{1}{\mu\epsilon} \nabla \times \nabla \times \mathbf{V} + \tau^2 \mathbf{V} + \mathbf{f}(x, \tau) = \mathbf{0} \quad \text{in } \mathbf{R}^3. \quad (1.14)$$

Define another indicator function

$$\tilde{I}_{\mathbf{f}}(\tau, T) = \int_B \mathbf{f}(x, \tau) \cdot (\mathbf{W}_e - \mathbf{V}_e^0) dx. \quad (1.15)$$

The *theoretical advantage* of this indicator function compared with (1.5) is that one has no need of computing \mathbf{V}_e which requires the space time computation of \mathbf{E}_0 and \mathbf{H}_0 on B over time interval $]0, T[$. Instead, in (1.15) one can compute \mathbf{V}_e^0 on B in advance by solving only (1.14). Our result on this indicator function is the following.

Theorem 1.2. *All the statements of Theorem 1 are valid if $I_{\mathbf{f}_j}(\tau, T)$ is replaced with $\tilde{I}_{\mathbf{f}_j}(\tau, T)$.*

A brief outline of this paper is as follows. Theorems 1.1 and 1.2 are proven in Section 2 by using Lemmas 2.1-2.3. Lemma 2.1 gives lower and upper estimates for $I_{\mathbf{f}}(\tau)$ as $\tau \rightarrow \infty$ with a single \mathbf{f} and the proof is described in Section 3. The proof is based on a *rough* asymptotic formula of $I_{\mathbf{f}}(\tau)$ as $\tau \rightarrow \infty$ which is proved in Subsection 3.2. Lemma 2.2 is concerned with an estimation of the sum of the two indicator functions corresponding to two input current sources in the case when λ satisfies (A.I) or (A.II). It is proved in Section 4. The statements (ii) and (iii) in Theorem 1.1 are direct consequences of Lemmas 2.1 and 2.2. Lemma 2.3 describes a simple estimate of the absolute value of $I_{\mathbf{f}}(\tau)$ as $\tau \rightarrow \infty$ which needs for the proof of (i) and (1.12) in Theorem 1.1. The proof is given in Subsection 4.5. The final section is devoted to conclusions and some of problems to be solved in the future.

2 Proof of Theorems 1.1 and 1.2

One can obtain immediately the validity of the statements (ii) and (iii) in Theorem 1.1 from the following two lemmas.

Lemma 2.1. *We have, as $\tau \rightarrow \infty$*

$$\tilde{J}_e(\tau) + O(\tau^{-1}e^{-\tau T}) \leq I_{\mathbf{f}}(\tau, T) \quad (2.1)$$

and

$$I_{\mathbf{f}}(\tau, T) \leq \tilde{J}_e(\tau) + \frac{\tau}{\epsilon} \int_{\partial D} \lambda \left| \boldsymbol{\nu} \times (\mathbf{V}_e \times \boldsymbol{\nu}) + \frac{1}{\lambda} \mathbf{V}_m \times \boldsymbol{\nu} \right|^2 dS + O(\tau^{-1}e^{-\tau T}), \quad (2.2)$$

where

$$\mathbf{V}_m = \mathbf{V}_m(x, \tau) = \int_0^T e^{-\tau t} \mathbf{H}_0(x, \tau) dt \quad (2.3)$$

and

$$\tilde{J}_e(\tau) = \frac{1}{\mu\epsilon} \int_{\partial D} (\boldsymbol{\nu} \times \mathbf{V}_e) \cdot \nabla \times \mathbf{V}_e dS - \frac{\tau}{\epsilon} \int_{\partial D} \frac{1}{\lambda} |\mathbf{V}_m \times \boldsymbol{\nu}|^2 dS. \quad (2.4)$$

Let $\mathbf{V}_{e,j}$ and $\mathbf{V}_{m,j}$ with $j = 1, 2$ denote \mathbf{V}_e and \mathbf{V}_m given by (1.8) and (2.3) using \mathbf{E}_0 and \mathbf{H}_0 with $\mathbf{J} = \mathbf{J}_j$.

Lemma 2.2. *Let $\tilde{J}_{e,j}$ with $j = 1, 2$ denote the \tilde{J}_e given by (2.4) for $\mathbf{V}_e = \mathbf{V}_{e,j}$ and $\mathbf{V}_m = \mathbf{V}_{m,j}$.*

(i) *If λ satisfies (A.I), then there exist positive numbers ρ , C' and τ_0 such that, for all $\tau \geq \tau_0$*

$$\sum_{j=1}^2 \tilde{J}_{e,j}(\tau) \geq C' \tau^{-\rho} e^{-2\tau\sqrt{\mu\epsilon} \text{dist}(D,B)} + O(\tau^{-1/2} e^{-\tau T}).$$

(ii) If λ satisfies (A.II), then there exist positive numbers ρ , C' and τ_0 such that, for all $\tau \geq \tau_0$

$$\begin{aligned} & \sum_{j=1}^2 \left(\tilde{J}_{e,j}(\tau) + \frac{\tau}{\epsilon} \int_{\partial D} \lambda \left| \boldsymbol{\nu} \times (\mathbf{V}_{e,j} \times \boldsymbol{\nu}) + \frac{1}{\lambda} \mathbf{V}_{m,j} \times \boldsymbol{\nu} \right|^2 dS \right) \\ & \leq -C' \tau^{-\rho} e^{-2\tau\sqrt{\mu\epsilon} \text{dist}(D,B)} + O(\tau^{-1/2} e^{-\tau T}). \end{aligned}$$

For the proof of (i) and formula (1.12) in Theorem 1.1 we need the following lemma.

Lemma 2.3. *We have, as $\tau \rightarrow \infty$*

$$I_{\mathbf{f}}(\tau, T) = O\left(\tau^{-1} e^{2\tau\sqrt{\mu\epsilon}\eta} (\tilde{f}(\tau))^2 \int_{\partial D} v^2 dS\right) + O(\tau^{-1/2} e^{-\tau T}). \quad (2.5)$$

Since $\tilde{f}(\tau) = O(\tau^{-3/2})$, from (2.5) we have, as $\tau \rightarrow \infty$

$$e^{\tau T} I_{\mathbf{f}}(\tau, T) = O(\tau^{-4} e^{\tau(T-2\sqrt{\mu\epsilon} \text{dist}(D,B))}) + O(\tau^{-1/2}).$$

This yields (i) of Theorem 1.1. Furthermore a combination of this and Lemma 2.1 gives (1.12). Note that ρ in (i) and (ii) of Lemma 2.2 can be chosen as $\rho = 2\gamma + 5$. See the end of the proof of (i) in Subsection 4.3. Note that γ in (1.11) has to be $\gamma \geq 3/2$ by the asymptotics of $\tilde{f}(\tau)$ mentioned above.

Theorem 1.2 is a transplantation of Theorem 1.1 via the following simple estimate:

$$\|\mathbf{V}_e - \mathbf{V}_e^0\|_{L^2(\mathbf{R}^3)} = O(\tau^{-1} e^{-\tau T}). \quad (2.6)$$

See (3.40) in Subsection 3.2. Since it is easy to see that $\tilde{f}(\tau) = O(\tau^{-3/2})$ and hence $\|\mathbf{f}(\cdot, \tau)\|_{L^2(B)} = O(\tau^{-1/2})$. This together with (2.6) gives

$$\tilde{I}_{\mathbf{f}}(\tau, T) = I_{\mathbf{f}}(\tau, T) + O(\tau^{-3/2} e^{-\tau T}).$$

Now from this and Lemmas 2.1-2.3 we obtain Theorem 1.2.

Thus everything is reduced to giving the proof of Lemmas 2.1-2.3.

3 Proof of Lemma 2.1

3.1 Preliminaries

Define

$$\mathbf{W}_m(x, \tau) = \int_0^T e^{-\tau t} \mathbf{H}(x, t) dt.$$

From (1.1) and (1.2) we see that \mathbf{W}_e and \mathbf{W}_m satisfy

$$\begin{cases} \nabla \times \mathbf{W}_e + \tau \mu \mathbf{W}_m = -e^{-\tau T} \mu \mathbf{H}(x, T) & \text{in } \mathbf{R}^3 \setminus \overline{D}, \\ \nabla \times \mathbf{W}_m - \tau \epsilon \mathbf{W}_e - \frac{\epsilon}{\tau} \mathbf{f}(x, \tau) = e^{-\tau T} \epsilon \mathbf{E}(x, T) & \text{in } \mathbf{R}^3 \setminus \overline{D} \end{cases} \quad (3.1)$$

and

$$\boldsymbol{\nu} \times \mathbf{W}_m - \lambda \boldsymbol{\nu} \times (\mathbf{W}_e \times \boldsymbol{\nu}) = \mathbf{0} \quad \text{on } \partial D. \quad (3.2)$$

Note that (3.2) is equivalent to

$$\boldsymbol{\nu} \times (\mathbf{W}_m \times \boldsymbol{\nu}) + \lambda \boldsymbol{\nu} \times \mathbf{W}_e = \mathbf{0} \quad \text{on } \partial D. \quad (3.3)$$

Taking the rotation of equations on (3.1), we obtain

$$\begin{cases} \frac{1}{\mu\epsilon} \nabla \times \nabla \times \mathbf{W}_e + \tau^2 \mathbf{W}_e + \mathbf{f}(x, \tau) = e^{-\tau T} \mathbf{F}_e(x, \tau) & \text{in } \mathbf{R}^3 \setminus \overline{D}, \\ \frac{1}{\mu\epsilon} \nabla \times \nabla \times \mathbf{W}_m + \tau^2 \mathbf{W}_m - \frac{1}{\tau\mu} \nabla \times \mathbf{f}(x, \tau) = e^{-\tau T} \mathbf{F}_m(x, \tau) & \text{in } \mathbf{R}^3 \setminus \overline{D}, \end{cases} \quad (3.4)$$

where

$$\begin{cases} \mathbf{F}_e(x, \tau) = - \left(\tau \mathbf{E}(x, T) + \frac{1}{\epsilon} \nabla \times \mathbf{H}(x, T) \right), \\ \mathbf{F}_m(x, \tau) = - \left(\tau \mathbf{H}(x, T) - \frac{1}{\mu} \nabla \times \mathbf{E}(x, T) \right). \end{cases} \quad (3.5)$$

Vector valued functions \mathbf{V}_e and \mathbf{V}_m satisfy

$$\begin{cases} \nabla \times \mathbf{V}_m - \tau\epsilon \mathbf{V}_e - \frac{\epsilon}{\tau} \mathbf{f} = e^{-\tau T} \epsilon \mathbf{E}_0(x, T) & \text{in } \mathbf{R}^3, \\ \nabla \times \mathbf{V}_e + \tau\mu \mathbf{V}_m = -e^{-\tau T} \mu \mathbf{H}_0(x, T) & \text{in } \mathbf{R}^3. \end{cases} \quad (3.6)$$

Taking the rotation of equations on (3.6), we obtain

$$\begin{cases} \frac{1}{\mu\epsilon} \nabla \times \nabla \times \mathbf{V}_e + \tau^2 \mathbf{V}_e + \mathbf{f}(x, \tau) = e^{-\tau T} \mathbf{F}_e^0(x, \tau) & \text{in } \mathbf{R}^3, \\ \frac{1}{\mu\epsilon} \nabla \times \nabla \times \mathbf{V}_m + \tau^2 \mathbf{V}_m - \frac{1}{\tau\mu} \nabla \times \mathbf{f}(x, \tau) = e^{-\tau T} \mathbf{F}_m^0(x, \tau) & \text{in } \mathbf{R}^3, \end{cases} \quad (3.7)$$

where

$$\begin{cases} \mathbf{F}_e^0(x, \tau) = - \left(\tau \mathbf{E}_0(x, T) + \frac{1}{\epsilon} \nabla \times \mathbf{H}_0(x, T) \right), \\ \mathbf{F}_m^0(x, \tau) = - \left(\tau \mathbf{H}_0(x, T) - \frac{1}{\mu} \nabla \times \mathbf{E}_0(x, T) \right). \end{cases} \quad (3.8)$$

Define

$$\begin{cases} \mathbf{R}_e = \mathbf{W}_e - \mathbf{V}_e, \\ \mathbf{R}_m = \mathbf{W}_m - \mathbf{V}_m. \end{cases}$$

From (3.1) and (3.6) we see that \mathbf{R}_e and \mathbf{R}_m satisfy

$$\begin{cases} \nabla \times \mathbf{R}_m - \tau\epsilon \mathbf{R}_e = e^{-\tau T} \epsilon \mathbf{F} & \text{in } \mathbf{R}^3 \setminus \overline{D}, \\ \nabla \times \mathbf{R}_e + \tau\mu \mathbf{R}_m = -e^{-\tau T} \mu \mathbf{G} & \text{in } \mathbf{R}^3 \setminus \overline{D}, \end{cases} \quad (3.9)$$

where

$$\begin{cases} \mathbf{F} = \mathbf{E}(x, T) - \mathbf{E}_0(x, T), \\ \mathbf{G} = \mathbf{H}(x, T) - \mathbf{H}_0(x, T). \end{cases} \quad (3.10)$$

Taking the differences of (3.4) from (3.7), we see that \mathbf{R}_\star with $\star = e, m$ satisfy

$$\frac{1}{\mu\epsilon} \nabla \times \nabla \times \mathbf{R}_\star + \tau^2 \mathbf{R}_\star = e^{-\tau T} (\mathbf{F}_\star(x, \tau) - \mathbf{F}_\star^0(x, \tau)) \quad \text{in } \mathbf{R}^3 \setminus \overline{D}. \quad (3.11)$$

Define

$$E_e(\tau) = \frac{1}{\epsilon\mu} \int_{\mathbf{R}^3 \setminus \overline{D}} |\nabla \times \mathbf{R}_e|^2 dx + \tau^2 \int_{\mathbf{R}^3 \setminus \overline{D}} |\mathbf{R}_e|^2 dx. \quad (3.12)$$

3.2 Rough asymptotic formula of the indicator function

We start with having the following asymptotic formula of the indicator function.

Proposition 3.1. *It holds that, as $\tau \rightarrow \infty$*

$$\int_{\mathbf{R}^3 \setminus \overline{D}} \mathbf{f}(x, \tau) \cdot \mathbf{R}_e dx = \tilde{J}_e(\tau) + \tilde{E}_e(\tau) + O(\tau^{-1} e^{-\tau T}), \quad (3.13)$$

where $\tilde{J}_e(\tau)$ is given by (2.4) and

$$\tilde{E}_e(\tau) = E_e(\tau) + \frac{\tau}{\epsilon} \int_{\partial D} \frac{1}{\lambda} |\mathbf{R}_m \times \boldsymbol{\nu}|^2 dS. \quad (3.14)$$

Proof. The proof is divided into three steps.

Step 1. First we show that

$$\int_{\mathbf{R}^3 \setminus \overline{D}} \mathbf{f}(x, \tau) \cdot \mathbf{R}_e dx = \tilde{J}_e(\tau) + \tilde{E}_e(\tau) + e^{-\tau T} (R_1(\tau) - R_2(\tau)), \quad (3.15)$$

where

$$R_1(\tau) = \frac{1}{\epsilon} \int_{\partial D} \frac{1}{\lambda} (\mathbf{G} - \mathbf{H}_0) \cdot (\mathbf{V}_m \times \boldsymbol{\nu} + \mathbf{R}_m \times \boldsymbol{\nu}) dS \quad (3.16)$$

and

$$R_2(\tau) = \int_{\mathbf{R}^3 \setminus \overline{D}} \left\{ \mathbf{F}_e(x, \tau) \cdot \mathbf{R}_e - (\mathbf{F}_e(x, \tau) - \mathbf{F}_e^0(x, \tau)) \cdot \mathbf{V}_e \right\} dx. \quad (3.17)$$

This is proved as follows. Integration by parts gives

$$\begin{aligned} & \int_{\mathbf{R}^3 \setminus \overline{D}} \{ (\nabla \times \nabla \times \mathbf{W}_e) \cdot \mathbf{V}_e - (\nabla \times \nabla \times \mathbf{V}_e) \cdot \mathbf{W}_e \} dx \\ &= \int_{\partial D} \{ (\boldsymbol{\nu} \times (\nabla \times \mathbf{V}_e)) \cdot \mathbf{W}_e - (\boldsymbol{\nu} \times (\nabla \times \mathbf{W}_e)) \cdot \mathbf{V}_e \} dS. \end{aligned}$$

We have

$$(\boldsymbol{\nu} \times (\nabla \times \mathbf{V}_e)) \cdot \mathbf{W}_e = (\mathbf{W}_e \times \boldsymbol{\nu}) \cdot \nabla \times \mathbf{V}_e = -(\boldsymbol{\nu} \times \mathbf{W}_e) \cdot \nabla \times \mathbf{V}_e$$

and

$$\begin{aligned}
(\boldsymbol{\nu} \times (\nabla \times \mathbf{W}_e)) \cdot \mathbf{V}_e &= (\nabla \times \mathbf{W}_e) \times \mathbf{V}_e \cdot \boldsymbol{\nu} \\
&= (\mathbf{V}_e \times \boldsymbol{\nu}) \cdot (\nabla \times \mathbf{W}_e) \\
&= -(\boldsymbol{\nu} \times \mathbf{V}_e) \cdot (\nabla \times \mathbf{W}_e).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{\mathbf{R}^3 \setminus \overline{D}} \{(\nabla \times \nabla \times \mathbf{W}_e) \cdot \mathbf{V}_e - (\nabla \times \nabla \times \mathbf{V}_e) \cdot \mathbf{W}_e\} dx \\
&= \int_{\partial D} (-(\boldsymbol{\nu} \times \mathbf{W}_e) \cdot \nabla \times \mathbf{V}_e + (\boldsymbol{\nu} \times \mathbf{V}_e) \cdot (\nabla \times \mathbf{W}_e)) dS.
\end{aligned}$$

Substituting the first equations on (3.4) and (3.7) into this left-hand side, we obtain

$$\begin{aligned}
&\frac{1}{\mu\epsilon} \int_{\partial D} (-(\boldsymbol{\nu} \times \mathbf{W}_e) \cdot \nabla \times \mathbf{V}_e + (\boldsymbol{\nu} \times \mathbf{V}_e) \cdot \nabla \times \mathbf{W}_e) dS \\
&= \int_{\mathbf{R}^3 \setminus \overline{D}} \mathbf{f}(x, \tau) \cdot \mathbf{R}_e dx \\
&\quad + e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} \{(\mathbf{F}_e(x, \tau) - \mathbf{F}_e^0(x, \tau)) \cdot \mathbf{V}_e - \mathbf{F}_e^0(x, \tau) \cdot \mathbf{R}_e\} dx.
\end{aligned} \tag{3.18}$$

Using $\mathbf{W}_e = \mathbf{V}_e + \mathbf{R}_e$, one has

$$\begin{aligned}
&\int_{\partial D} (-(\boldsymbol{\nu} \times \mathbf{W}_e) \cdot \nabla \times \mathbf{V}_e + (\boldsymbol{\nu} \times \mathbf{V}_e) \cdot \nabla \times \mathbf{W}_e) dS \\
&= \int_{\partial D} (-(\boldsymbol{\nu} \times \mathbf{R}_e) \cdot \nabla \times \mathbf{V}_e + (\boldsymbol{\nu} \times \mathbf{V}_e) \cdot \nabla \times \mathbf{R}_e) dS.
\end{aligned} \tag{3.19}$$

Note that from (3.3), we have

$$\boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) + \lambda \boldsymbol{\nu} \times \mathbf{R}_e = -\boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) - \lambda \boldsymbol{\nu} \times \mathbf{V}_e \quad \text{on } \partial D, \tag{3.20}$$

and thus

$$\boldsymbol{\nu} \times \mathbf{V}_e = -\boldsymbol{\nu} \times \mathbf{R}_e - \frac{1}{\lambda} \{\boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) + \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu})\} \quad \text{on } \partial D$$

or equivalently,

$$\boldsymbol{\nu} \times \mathbf{R}_e = -\boldsymbol{\nu} \times \mathbf{V}_e - \frac{1}{\lambda} \{\boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) + \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu})\} \quad \text{on } \partial D.$$

From these we obtain

$$\begin{aligned}
&(\boldsymbol{\nu} \times \mathbf{V}_e) \cdot \nabla \times \mathbf{R}_e = -\boldsymbol{\nu} \times \mathbf{R}_e \cdot \nabla \times \mathbf{R}_e \\
&\quad - \frac{1}{\lambda} \{\boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) + \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu})\} \cdot \nabla \times \mathbf{R}_e \quad \text{on } \partial D
\end{aligned}$$

and

$$\begin{aligned}
& (\boldsymbol{\nu} \times \mathbf{R}_e) \cdot \nabla \times \mathbf{V}_e = -\boldsymbol{\nu} \times \mathbf{V}_e \cdot \nabla \times \mathbf{V}_e \\
& -\frac{1}{\lambda} \{ \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) + \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) \} \cdot \nabla \times \mathbf{V}_e \quad \text{on } \partial D.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
& -(\boldsymbol{\nu} \times \mathbf{R}_e) \cdot \nabla \times \mathbf{V}_e + (\boldsymbol{\nu} \times \mathbf{V}_e) \cdot \nabla \times \mathbf{R}_e \\
& = \boldsymbol{\nu} \times \mathbf{V}_e \cdot \nabla \times \mathbf{V}_e - \boldsymbol{\nu} \times \mathbf{R}_e \cdot \nabla \times \mathbf{R}_e \\
& + \frac{1}{\lambda} \{ \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) + \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) \} \cdot \nabla \times \mathbf{V}_e \\
& - \frac{1}{\lambda} \{ \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) + \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) \} \cdot \nabla \times \mathbf{R}_e.
\end{aligned} \tag{3.21}$$

Now from (3.18), (3.19) and (3.21) we obtain

$$\begin{aligned}
& \int_{\mathbf{R}^3 \setminus \overline{D}} \mathbf{f}(x, \tau) \cdot \mathbf{R}_e \, dx \\
& = \frac{1}{\mu\epsilon} \int_{\partial D} (\boldsymbol{\nu} \times \mathbf{V}_e \cdot \nabla \times \mathbf{V}_e - \boldsymbol{\nu} \times \mathbf{R}_e \cdot \nabla \times \mathbf{R}_e) \, dx \\
& + \frac{1}{\mu\epsilon} \int_{\partial D} \frac{1}{\lambda} \{ \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) + \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) \} \cdot \nabla \times \mathbf{V}_e \, dS \\
& - \frac{1}{\mu\epsilon} \int_{\partial D} \frac{1}{\lambda} \{ \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) + \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) \} \cdot \nabla \times \mathbf{R}_e \, dS \\
& - e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} \{ (\mathbf{F}_e(x, \tau) - \mathbf{F}_e^0(x, \tau)) \cdot \mathbf{V}_e - \mathbf{F}_e^0(x, \tau) \cdot \mathbf{R}_e \} \, dx.
\end{aligned} \tag{3.22}$$

(3.11) with $\star = e$ and (3.12) and integration by parts give

$$\begin{aligned}
& e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} (\mathbf{F}_e(x, \tau) - \mathbf{F}_e^0(x, \tau)) \cdot \mathbf{R}_e \, dx \\
& = -\frac{1}{\mu\epsilon} \int_{\partial D} \boldsymbol{\nu} \times (\nabla \times \mathbf{R}_e) \cdot \mathbf{R}_e \, dS + E_e(\tau) \\
& = \frac{1}{\mu\epsilon} \int_{\partial D} (\boldsymbol{\nu} \times \mathbf{R}_e) \cdot \nabla \times \mathbf{R}_e \, dS + E_e(\tau),
\end{aligned}$$

that is,

$$\begin{aligned}
& -\frac{1}{\mu\epsilon} \int_{\partial D} (\boldsymbol{\nu} \times \mathbf{R}_e) \cdot \nabla \times \mathbf{R}_e \, dS \\
& = E_e(\tau) - e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} (\mathbf{F}_e(x, \tau) - \mathbf{F}_e^0(x, \tau)) \cdot \mathbf{R}_e \, dx.
\end{aligned} \tag{3.23}$$

Define

$$J_e(\tau) = \frac{1}{\epsilon\mu} \int_D |\nabla \times \mathbf{V}_e|^2 dx + \tau^2 \int_D |\mathbf{V}_e|^2 dx. \quad (3.24)$$

We have also from the first equation on (3.7) and (3.24)

$$\frac{1}{\mu\epsilon} \int_{\partial D} (\boldsymbol{\nu} \times \mathbf{V}_e) \cdot \nabla \times \mathbf{V}_e dS = J_e(\tau) - e^{-\tau T} \int_D \mathbf{F}_e^0(x, \tau) \cdot \mathbf{V}_e dx. \quad (3.25)$$

Now from (2.19), (2.20), (3.22) and (3.25), we obtain

$$\begin{aligned} & \int_{\mathbf{R}^3 \setminus \overline{D}} \mathbf{f}(x, \tau) \cdot \mathbf{R}_e dx \\ &= J_e(\tau) - e^{-\tau T} \int_D \mathbf{F}_e^0(x, \tau) \cdot \mathbf{V}_e dx + E_e(\tau) \\ &+ \frac{1}{\mu\epsilon} \int_{\partial D} \frac{1}{\lambda} \{ \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) \cdot \nabla \times \mathbf{V}_e - \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \cdot \nabla \times \mathbf{R}_e \} dS \\ &+ \frac{1}{\mu\epsilon} \int_{\partial D} \frac{1}{\lambda} \{ -\boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) \cdot \nabla \times \mathbf{R}_e + (\boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu})) \cdot \nabla \times \mathbf{V}_e \} dS \\ &- e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} \{ \mathbf{F}_e(x, \tau) \cdot \mathbf{R}_e - (\mathbf{F}_e(x, \tau) - \mathbf{F}_e^0(x, \tau)) \cdot \mathbf{V}_e \} dx. \end{aligned} \quad (3.26)$$

Here we make an order of the integrals over ∂D in (3.26). It follows from the second equation on (3.6) that

$$\begin{aligned} & \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) \cdot \nabla \times \mathbf{V}_e \\ &= (\nabla \times \mathbf{V}_e) \times \boldsymbol{\nu} \cdot (\mathbf{R}_m \times \boldsymbol{\nu}) \\ &= -\tau\mu \mathbf{V}_m \times \boldsymbol{\nu} \cdot \mathbf{R}_m \times \boldsymbol{\nu} - e^{-\tau T} \mu \mathbf{H}_0 \cdot (\mathbf{R}_m \times \boldsymbol{\nu}) \end{aligned} \quad (3.27)$$

and

$$\begin{aligned} & (\boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu})) \cdot \nabla \times \mathbf{V}_e \\ &= (\nabla \times \mathbf{V}_e) \times \boldsymbol{\nu} \cdot (\mathbf{V}_m \times \boldsymbol{\nu}) \\ &= -\tau\mu |\mathbf{V}_m \times \boldsymbol{\nu}|^2 - e^{-\tau T} \mu \mathbf{H}_0 \cdot (\mathbf{V}_m \times \boldsymbol{\nu}). \end{aligned} \quad (3.28)$$

It follows from the second equation on (3.9) that

$$\begin{aligned} & \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \cdot \nabla \times \mathbf{R}_e \\ &= (\nabla \times \mathbf{R}_e) \times \boldsymbol{\nu} \cdot (\mathbf{V}_m \times \boldsymbol{\nu}) \\ &= -\tau\mu \mathbf{R}_m \times \boldsymbol{\nu} \cdot \mathbf{V}_m \times \boldsymbol{\nu} - e^{-\tau T} \mu \mathbf{G} \cdot (\mathbf{V}_m \times \boldsymbol{\nu}) \end{aligned} \quad (3.29)$$

and

$$\begin{aligned}
& -(\boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu})) \cdot \nabla \times \mathbf{R}_e \\
& = -(\nabla \times \mathbf{R}_e) \times \boldsymbol{\nu} \cdot (\mathbf{R}_m \times \boldsymbol{\nu}) \\
& = \tau \mu |\mathbf{R}_m \times \boldsymbol{\nu}|^2 + e^{-\tau T} \mu \mathbf{G} \cdot (\mathbf{R}_m \times \boldsymbol{\nu}).
\end{aligned} \tag{3.30}$$

A combination of (3.27) and (3.29) gives

$$\begin{aligned}
& \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) \cdot \nabla \times \mathbf{V}_e - \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \cdot \nabla \times \mathbf{R}_e \\
& = e^{-\tau T} \mu \{ \mathbf{G} \cdot (\mathbf{V}_m \times \boldsymbol{\nu}) - \mathbf{H}_0 \cdot (\mathbf{R}_m \times \boldsymbol{\nu}) \}.
\end{aligned} \tag{3.31}$$

A combination of (3.28) and (3.30) gives

$$\begin{aligned}
& -(\boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu})) \cdot \nabla \times \mathbf{R}_e + (\boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu})) \cdot \nabla \times \mathbf{V}_e \\
& = -\tau \mu |\mathbf{V}_m \times \boldsymbol{\nu}|^2 + \tau \mu |\mathbf{R}_m \times \boldsymbol{\nu}|^2 + e^{-\tau T} \mu \{ \mathbf{G} \cdot (\mathbf{R}_m \times \boldsymbol{\nu}) - \mathbf{H}_0 \cdot (\mathbf{V}_m \times \boldsymbol{\nu}) \}.
\end{aligned} \tag{3.32}$$

It follows from (2.4) and (3.25) that

$$\tilde{J}_e(\tau) = J_e(\tau) - \frac{\tau}{\epsilon} \int_{\partial D} \frac{1}{\lambda} |\mathbf{V}_m \times \boldsymbol{\nu}|^2 dS - e^{-\tau T} \int_D \mathbf{F}_e^0(x, \tau) \cdot \mathbf{V}_e dx. \tag{3.33}$$

Now substituting (3.31), (3.32) and (3.33) into (3.26), we obtain (3.15).

Step 2. We have, as $\rightarrow \infty$

$$\|\mathbf{V}_e\|_{L^2(\mathbf{R}^3)} = O(\tau^{-5/2}); \tag{3.34}$$

for $\star = e, m$

$$\|\mathbf{V}_\star \times \boldsymbol{\nu}\|_{L^2(\partial D)} = O(\tau^{-3/2}). \tag{3.35}$$

This is proved as follows.

Recall \mathbf{V}_e^0 is the weak solution of (1.14). Since $\|\mathbf{f}(\cdot, \tau)\|_{L^2(B)} = O(\tau^{-1/2})$, it is easy to see that we have

$$\|\mathbf{V}_e^0\|_{L^2(\mathbf{R}^3)} = O(\tau^{-5/2}). \tag{3.36}$$

and

$$\|\nabla \times \mathbf{V}_e^0\|_{L^2(\mathbf{R}^3)} = O(\tau^{-3/2}). \tag{3.37}$$

Next define

$$\mathbf{Z} = \mathbf{V}_e - \mathbf{V}_e^0.$$

Then from the first equation on (3.7) and (1.14) we have

$$\frac{1}{\mu \epsilon} \nabla \times \nabla \times \mathbf{Z} + \tau^2 \mathbf{Z} = e^{-\tau T} \mathbf{F}_e^0 \quad \text{in } \mathbf{R}^n. \tag{3.38}$$

This gives

$$\frac{1}{\mu \epsilon} \int_{\mathbf{R}^n} |\nabla \times \mathbf{Z}|^2 dx + \tau^2 \int_{\mathbf{R}^3} |\mathbf{Z}|^2 dx = e^{-\tau T} \int_{\mathbf{R}^3} \mathbf{F}_e^0 \cdot \mathbf{Z} dx$$

and hence

$$\frac{1}{\mu \epsilon} \int_{\mathbf{R}^3} |\nabla \times \mathbf{Z}|^2 dx + \tau^2 \int_{\mathbf{R}^3} |\mathbf{Z}|^2 dx \leq \frac{e^{-2\tau T}}{\tau^2} \int_{\mathbf{R}^3} |\mathbf{F}_e^0|^2 dx.$$

Then applying the first equation on (3.8) to this right-hand side, we conclude that

$$\|\mathbf{Z}\|_{L^2(\mathbf{R}^3)} = O(\tau^{-1}e^{-\tau T}), \quad (3.39)$$

and

$$\|\nabla \times \mathbf{Z}\|_{L^2(\mathbf{R}^3)} = O(e^{-\tau T}). \quad (3.40)$$

Now a combination of (3.36) and (3.39) gives (3.34). Note also that a combination of (3.37) and (3.40) gives

$$\|\nabla \times \mathbf{V}_e\|_{L^2(\mathbf{R}^3)} = O(\tau^{-3/2}). \quad (3.41)$$

Then, the trace theorem [5] yields (3.35) with $\star = e$. Moreover, using equations (3.6) together with (3.34) and (3.41), we obtain

$$\|\mathbf{V}_m\|_{L^2(\mathbf{R}^3)} = O(\tau^{-5/2}),$$

and

$$\|\nabla \times \mathbf{V}_m\|_{L^2(\mathbf{R}^3)} = O(\tau^{-3/2}).$$

Then, the trace theorem [5] yields (3.35) with $\star = m$.

Step 3. We have, as $\tau \rightarrow \infty$

$$\|\mathbf{R}_e\|_{L^2(\mathbf{R}^3 \setminus \overline{D})} = O(\tau^{-2}); \quad (3.42)$$

$$\|\mathbf{R}_m \times \boldsymbol{\nu}\|_{L^2(\partial D)} = O(\tau^{-3/2}). \quad (3.43)$$

This is proved as follows.

From (3.9) we obtain

$$\begin{aligned} & \int_{\mathbf{R}^3 \setminus \overline{D}} \nabla \cdot (\mathbf{R}_e \times \mathbf{R}_m) dx + \tau \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon |\mathbf{R}_e|^2 + \mu |\mathbf{R}_m|^2) dx \\ &= -e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon \mathbf{F} \cdot \mathbf{R}_e + \mu \mathbf{G} \cdot \mathbf{R}_m) dx. \end{aligned} \quad (3.44)$$

Here we note that

$$\int_{\mathbf{R}^3 \setminus \overline{D}} \nabla \cdot (\mathbf{R}_e \times \mathbf{R}_m) dx = - \int_{\partial D} \boldsymbol{\nu} \cdot \mathbf{R}_e \times \mathbf{R}_m dS$$

and from (3.20) we have

$$\begin{aligned} & \boldsymbol{\nu} \cdot \mathbf{R}_e \times \mathbf{R}_m \\ &= \mathbf{R}_m \cdot (\boldsymbol{\nu} \times \mathbf{R}_e) \\ &= \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) \cdot (\boldsymbol{\nu} \times \mathbf{R}_e) \\ &= -\lambda \boldsymbol{\nu} \times \mathbf{R}_e \cdot \boldsymbol{\nu} \times \mathbf{R}_e - \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \cdot \boldsymbol{\nu} \times \mathbf{R}_e - \lambda \boldsymbol{\nu} \times \mathbf{V}_e \cdot \boldsymbol{\nu} \times \mathbf{R}_e \\ &= -\lambda |\boldsymbol{\nu} \times \mathbf{R}_e|^2 - (\boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) + \lambda \boldsymbol{\nu} \times \mathbf{V}_e) \cdot \boldsymbol{\nu} \times \mathbf{R}_e \\ &= -\lambda \left| \boldsymbol{\nu} \times \mathbf{R}_e + \frac{1}{2} \left(\boldsymbol{\nu} \times \mathbf{V}_e + \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \right) \right|^2 \\ & \quad + \frac{\lambda}{4} \left| \boldsymbol{\nu} \times \mathbf{V}_e + \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \right|^2. \end{aligned} \quad (3.45)$$

Therefore (3.44) becomes

$$\begin{aligned}
& \int_{\partial D} \lambda \left| \boldsymbol{\nu} \times \mathbf{R}_e + \frac{1}{2} \left(\boldsymbol{\nu} \times \mathbf{V}_e + \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \right) \right|^2 dS + \tau \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon |\mathbf{R}_e|^2 + \mu |\mathbf{R}_m|^2) dx \\
& + e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon \mathbf{F} \cdot \mathbf{R}_e + \mu \mathbf{G} \cdot \mathbf{R}_m) dx \\
& = \frac{1}{4} \int_{\partial D} \lambda \left| \boldsymbol{\nu} \times \mathbf{V}_e + \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \right|^2 dS
\end{aligned}$$

and hence

$$\begin{aligned}
& \int_{\partial D} \lambda \left| \boldsymbol{\nu} \times \mathbf{R}_e + \frac{1}{2} \left(\boldsymbol{\nu} \times \mathbf{V}_e + \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \right) \right|^2 dS \\
& + \tau \int_{\mathbf{R}^3 \setminus \overline{D}} \left(\epsilon \left| \mathbf{R}_e + \frac{e^{-\tau T}}{2\tau} \mathbf{F} \right|^2 + \mu \left| \mathbf{R}_m + \frac{e^{-\tau T}}{2\tau} \mathbf{G} \right|^2 \right) dx \\
& = \frac{1}{4} \int_{\partial D} \lambda \left| \boldsymbol{\nu} \times \mathbf{V}_e + \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \right|^2 dS + \frac{e^{-2\tau T}}{4\tau} \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon |\mathbf{F}|^2 + \mu |\mathbf{G}|^2) dx.
\end{aligned}$$

This yields

$$\begin{aligned}
& \frac{1}{2} \int_{\partial D} \lambda |\boldsymbol{\nu} \times \mathbf{R}_e|^2 dS + \frac{1}{2} \tau \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon |\mathbf{R}_e|^2 + \mu |\mathbf{R}_m|^2) dx \\
& \leq \frac{1}{2} \int_{\partial D} \lambda \left| \boldsymbol{\nu} \times \mathbf{V}_e + \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \right|^2 dS + \frac{e^{-2\tau T}}{2\tau} \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon |\mathbf{F}|^2 + \mu |\mathbf{G}|^2) dx,
\end{aligned}$$

and hence

$$\begin{aligned}
& \lambda_0 \int_{\partial D} |\boldsymbol{\nu} \times \mathbf{R}_e|^2 dS + \tau \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon |\mathbf{R}_e|^2 + \mu |\mathbf{R}_m|^2) dx \\
& \leq \int_{\partial D} \lambda \left| \boldsymbol{\nu} \times \mathbf{V}_e + \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \right|^2 dS + \frac{e^{-2\tau T}}{\tau} \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon |\mathbf{F}|^2 + \mu |\mathbf{G}|^2) dx,
\end{aligned} \tag{3.46}$$

where $\lambda_0 = \min_{x \in \partial D} \lambda(x)$. Now applying (3.35) and (3.10) to the right-hand side on (3.46) we obtain (3.42).

The proof of estimate (3.43) is the following. First from (3.2) we have

$$\boldsymbol{\nu} \times (\mathbf{R}_e \times \boldsymbol{\nu}) = \frac{1}{\lambda} \boldsymbol{\nu} \times \mathbf{R}_m - \mathbf{V}_{em},$$

where

$$\mathbf{V}_{em} = \boldsymbol{\nu} \times (\mathbf{V}_e \times \boldsymbol{\nu}) + \frac{1}{\lambda} \mathbf{V}_m \times \boldsymbol{\nu}. \tag{3.47}$$

Thus we have another expression for (3.45):

$$\begin{aligned}
& \boldsymbol{\nu} \cdot \mathbf{R}_e \times \mathbf{R}_m \\
&= -\mathbf{R}_e \cdot (\boldsymbol{\nu} \times \mathbf{R}_m) \\
&= -\boldsymbol{\nu} \times (\mathbf{R}_e \times \boldsymbol{\nu}) \cdot (\boldsymbol{\nu} \times \mathbf{R}_m) \\
&= -\frac{1}{\lambda} |\boldsymbol{\nu} \times \mathbf{R}_m|^2 + \mathbf{V}_{em} \cdot \boldsymbol{\nu} \times \mathbf{R}_m \\
&= -\frac{1}{\lambda} \left| \boldsymbol{\nu} \times \mathbf{R}_m - \frac{\lambda}{2} \mathbf{V}_{em} \right|^2 + \frac{\lambda}{4} |\mathbf{V}_{em}|^2.
\end{aligned}$$

Therefore (3.44) becomes

$$\begin{aligned}
& \int_{\partial D} \frac{1}{\lambda} \left| \boldsymbol{\nu} \times \mathbf{R}_m - \frac{\lambda}{2} \mathbf{V}_{em} \right|^2 dS + \tau \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon |\mathbf{R}_e|^2 + \mu |\mathbf{R}_m|^2) dx \\
&+ e^{-\tau T} \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon \mathbf{F} \cdot \mathbf{R}_e + \mu \mathbf{G} \cdot \mathbf{R}_m) dx \\
&= \frac{1}{4} \int_{\partial D} \lambda |\mathbf{V}_{em}|^2 dS
\end{aligned}$$

and hence

$$\begin{aligned}
& \int_{\partial D} \frac{1}{\lambda} \left| \boldsymbol{\nu} \times \mathbf{R}_m - \frac{\lambda}{2} \mathbf{V}_{em} \right|^2 dS \\
&+ \tau \int_{\mathbf{R}^3 \setminus \overline{D}} \left(\epsilon \left| \mathbf{R}_e + \frac{e^{-\tau T}}{2\tau} \mathbf{F} \right|^2 + \mu \left| \mathbf{R}_m + \frac{e^{-\tau T}}{2\tau} \mathbf{G} \right|^2 \right) dx \\
&= \frac{1}{4} \int_{\partial D} \lambda |\mathbf{V}_{em}|^2 dS + \frac{e^{-2\tau T}}{4\tau} \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon |\mathbf{F}|^2 + \mu |\mathbf{G}|^2) dx.
\end{aligned}$$

This yields

$$\begin{aligned}
& \frac{1}{\lambda_1} \int_{\partial D} |\boldsymbol{\nu} \times \mathbf{R}_m|^2 dS + \tau \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon |\mathbf{R}_e|^2 + \mu |\mathbf{R}_m|^2) dx \\
&\leq \int_{\partial D} \lambda |\mathbf{V}_{em}|^2 dS + \frac{e^{-2\tau T}}{\tau} \int_{\mathbf{R}^3 \setminus \overline{D}} (\epsilon |\mathbf{F}|^2 + \mu |\mathbf{G}|^2) dx,
\end{aligned} \tag{3.48}$$

where $\lambda_1 = \max_{x \in \partial D} \lambda(x)$. Now applying (3.35) and (3.10) to the right-hand side on (3.48), we obtain (3.43).

Finally applying (3.35) and (3.43) to the right-hand side on (3.16), we obtain

$$R_1(\tau) = O(\tau^{-3/2}).$$

Applying (3.34) and (3.41) to the right-hand side on (3.17) and noting (3.5) and (3.8), we obtain

$$R_2(\tau) = O(\tau^{-1}) + O(\tau^{-3/2}) = O(\tau^{-1}).$$

□

Remark 3.1. Needless to say, estimates (3.35), (3.42) and (3.43) are quite rough. In fact, using the explicit expression of \mathbf{V}_e^0 outside B , we see that \mathbf{V}_e^0 together with its derivatives is exponentially decaying as $\tau \rightarrow \infty$ uniformly for all $x \in \overline{D}$. This yields better estimates than (3.35), (3.42) and (3.43). However, at this stage we do not need such a detailed information about \mathbf{V}_e^0 . This is a reason why we call (3.13) the *rough* asymptotic formula. Note that (3.39) is nothing but (2.6).

3.3 Finishing the proof of Lemma 2.1

In this subsection we derive the following estimate for $\tilde{E}_e(\tau)$ given by (3.14):

$$\tilde{E}_e(\tau) \leq \frac{\tau}{\epsilon} \int_{\partial D} \lambda |\mathbf{V}_{em}|^2 dS + O(e^{-2\tau T}), \quad (3.49)$$

where \mathbf{V}_{em} is given by (3.47). Note that from this together with (3.13) we obtain (2.1) and (2.2).

It follows from (3.12) and (3.23) that

$$\begin{aligned} & \frac{1}{\mu\epsilon} \int_{\mathbf{R}^3 \setminus \overline{D}} |\nabla \times \mathbf{R}_e|^2 dx + \tau^2 \int_{\mathbf{R}^3 \setminus \overline{D}} \left| \mathbf{R}_e - \frac{e^{-\tau T}}{2\tau^2} (\mathbf{F}_e - \mathbf{F}_e^0) \right|^2 dx \\ & + \frac{1}{\mu\epsilon} \int_{\partial D} (\boldsymbol{\nu} \times \mathbf{R}_e) \cdot \nabla \times \mathbf{R}_e dS \\ & = \frac{e^{-2\tau T}}{4\tau^2} \int_{\mathbf{R}^3 \setminus \overline{D}} |\mathbf{F}_e - \mathbf{F}_0|^2 dx. \end{aligned} \quad (3.50)$$

Since

$$\left| \mathbf{R}_e - \frac{e^{-\tau T}}{2\tau^2} (\mathbf{F}_e - \mathbf{F}_e^0) \right|^2 \geq \frac{1}{2} |\mathbf{R}_e|^2 - \frac{e^{-2\tau T}}{4\tau^4} |\mathbf{F}_e - \mathbf{F}_e^0|^2,$$

It follows from (3.50) that

$$\frac{1}{2} E_e(\tau) + \frac{1}{\mu\epsilon} \int_{\partial D} (\boldsymbol{\nu} \times \mathbf{R}_e) \cdot \nabla \times \mathbf{R}_e dS \leq \frac{e^{-2\tau T}}{2\tau^2} \int_{\mathbf{R}^3 \setminus \overline{D}} |\mathbf{F}_e - \mathbf{F}_0|^2 dx. \quad (3.51)$$

From (3.3) we have

$$\begin{aligned} & \boldsymbol{\nu} \times \mathbf{R}_e \\ & = \boldsymbol{\nu} \times (\mathbf{W}_e - \mathbf{V}_e) \\ & = -\frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{W}_m \times \boldsymbol{\nu}) - \boldsymbol{\nu} \times \mathbf{V}_e \\ & = -\frac{1}{\lambda} \boldsymbol{\nu} \times \{(\mathbf{W}_m - \mathbf{V}_m) \times \boldsymbol{\nu}\} - \boldsymbol{\nu} \times \mathbf{V}_e - \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \\ & = -\frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) - \boldsymbol{\nu} \times \mathbf{V}_e - \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}). \end{aligned}$$

Then, from the second equation on (3.9) together with this yields

$$\begin{aligned}
& \boldsymbol{\nu} \times \mathbf{R}_e \cdot \nabla \times \mathbf{R}_e \\
&= \left\{ \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) + \boldsymbol{\nu} \times \mathbf{V}_e + \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \right\} \cdot (\tau \mu \mathbf{R}_m + e^{-\tau T} \mu \mathbf{G}) \\
&= \frac{\tau \mu}{\lambda} \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) \cdot \mathbf{R}_m + \tau \mu \left\{ \boldsymbol{\nu} \times \mathbf{V}_e + \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \right\} \cdot \mathbf{R}_m \\
&\quad + e^{-\tau T} \mu \left\{ \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{R}_m \times \boldsymbol{\nu}) + \boldsymbol{\nu} \times \mathbf{V}_e + \frac{1}{\lambda} \boldsymbol{\nu} \times (\mathbf{V}_m \times \boldsymbol{\nu}) \right\} \cdot \mathbf{G} \\
&= \frac{\tau \mu}{\lambda} |\mathbf{R}_m \times \boldsymbol{\nu}|^2 + \tau \mu \left(\boldsymbol{\nu} \times (\mathbf{V}_e \times \boldsymbol{\nu}) + \frac{1}{\lambda} \mathbf{V}_m \times \boldsymbol{\nu} \right) \cdot \mathbf{R}_m \times \boldsymbol{\nu} \\
&\quad + e^{-\tau T} \mu \left(\frac{1}{\lambda} \mathbf{R}_m \times \boldsymbol{\nu} + \boldsymbol{\nu} \times (\mathbf{V}_e \times \boldsymbol{\nu}) + \frac{1}{\lambda} \mathbf{V}_m \times \boldsymbol{\nu} \right) \cdot \mathbf{G} \times \boldsymbol{\nu} \\
&= \frac{\tau \mu}{\lambda} |\mathbf{R}_m \times \boldsymbol{\nu}|^2 + \mu \left(\tau \mathbf{V}_{em} + \frac{e^{-\tau T}}{\lambda} \mathbf{G} \times \boldsymbol{\nu} \right) \cdot \mathbf{R}_m \times \boldsymbol{\nu} \\
&\quad + e^{-\tau T} \mu \mathbf{V}_{em} \cdot \mathbf{G} \times \boldsymbol{\nu}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
& \frac{1}{\mu \epsilon} \int_{\partial D} \boldsymbol{\nu} \times \mathbf{R}_e \cdot \nabla \times \mathbf{R}_e dS \\
&= \frac{\tau}{\epsilon} \int_{\partial D} \frac{1}{\lambda} |\mathbf{R}_m \times \boldsymbol{\nu}|^2 dS + \frac{\tau}{\epsilon} \int_{\partial D} \left(\mathbf{V}_{em} + \frac{e^{-\tau T}}{\tau \lambda} \mathbf{G} \times \boldsymbol{\nu} \right) \cdot \mathbf{R}_m \times \boldsymbol{\nu} dS \\
&\quad + \frac{e^{-\tau T}}{\epsilon} \int_{\partial D} \mathbf{V}_{em} \cdot \mathbf{G} \times \boldsymbol{\nu} dS \\
&= \frac{\tau}{\epsilon} \int_{\partial D} \frac{1}{\lambda} \left| \mathbf{R}_m \times \boldsymbol{\nu} + \frac{\lambda}{2} \left(\mathbf{V}_{em} + \frac{e^{-\tau T}}{\tau \lambda} \mathbf{G} \times \boldsymbol{\nu} \right) \right|^2 dS \tag{3.52} \\
&\quad - \frac{\tau}{4\epsilon} \int_{\partial D} \lambda \left| \mathbf{V}_{em} + \frac{e^{-\tau T}}{2\lambda} \mathbf{G} \times \boldsymbol{\nu} \right|^2 dS + \frac{e^{-\tau T}}{\epsilon} \int_{\partial D} \mathbf{V}_{em} \cdot \mathbf{G} \times \boldsymbol{\nu} dS \\
&\geq \frac{\tau}{2\epsilon} \int_{\partial D} \frac{1}{\lambda} |\mathbf{R}_m \times \boldsymbol{\nu}|^2 dS - \frac{\tau}{2\epsilon} \int_{\partial D} \lambda \left| \mathbf{V}_{em} + \frac{e^{-\tau T}}{\tau \lambda} \mathbf{G} \times \boldsymbol{\nu} \right|^2 dS \\
&\quad + \frac{e^{-\tau T}}{\epsilon} \int_{\partial D} \mathbf{V}_{em} \cdot \mathbf{G} \times \boldsymbol{\nu} dS.
\end{aligned}$$

Now a combination of (3.51) and (3.52) gives

$$\begin{aligned} \frac{1}{2}\tilde{E}_e(\tau) &\leq \frac{\tau}{2\epsilon} \int_{\partial D} \lambda \left| \mathbf{V}_{em} + \frac{e^{-\tau T}}{\tau\lambda} \mathbf{G} \times \boldsymbol{\nu} \right|^2 dS \\ &\quad + \frac{e^{-2\tau T}}{2\tau^2} \int_{\mathbf{R}^3 \setminus \bar{D}} |\mathbf{F}_e - \mathbf{F}_e^0|^2 dx - \frac{e^{-\tau T}}{\epsilon} \int_{\partial D} \mathbf{V}_{em} \cdot \mathbf{G} \times \boldsymbol{\nu} dS \end{aligned}$$

and hence

$$\begin{aligned} \tilde{E}_e(\tau) &\leq \frac{\tau}{\epsilon} \int_{\partial D} \lambda \left| \mathbf{V}_{em} + \frac{e^{-\tau T}}{\tau\lambda} \mathbf{G} \times \boldsymbol{\nu} \right|^2 dS \\ &\quad + \frac{e^{-2\tau T}}{\tau^2} \int_{\mathbf{R}^3 \setminus \bar{D}} |\mathbf{F}_e - \mathbf{F}_e^0|^2 dx - \frac{2e^{-\tau T}}{\epsilon} \int_{\partial D} \mathbf{V}_{em} \cdot \mathbf{G} \times \boldsymbol{\nu} dS \\ &= \frac{\tau}{\epsilon} \int_{\partial D} \lambda \left(|\mathbf{V}_{em}|^2 + \left| \frac{e^{-\tau T}}{\tau\lambda} \mathbf{G} \times \boldsymbol{\nu} \right|^2 \right) dS + \frac{e^{-2\tau T}}{\tau^2} \int_{\mathbf{R}^3 \setminus \bar{D}} |\mathbf{F}_e - \mathbf{F}_e^0|^2 dx. \end{aligned}$$

Then this together with (3.8) and (3.10) yields (3.49).

4 Proof of Lemmas 2.2 and 2.3

First we describe the proof of Lemma 2.2.

4.1 A reduction

Lemma 4.1 *We have*

$$\begin{aligned} \tilde{J}_e(\tau) &= \frac{1}{\mu\epsilon} \int_{\partial D} \frac{1}{c} \left\{ c \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) - (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} \right\} \cdot (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} dS \\ &\quad + O(\tau^{-1/2} e^{-\tau T}) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\tilde{J}_e(\tau) + \frac{1}{\mu\epsilon} \int_{\partial D} c |\mathbf{V}_{em}|^2 dS \\ &= \frac{1}{\mu\epsilon} \int_{\partial D} \left\{ c \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) - (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} \right\} \cdot \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) dS + O(\tau^{-1/2} e^{-\tau T}), \end{aligned} \tag{4.2}$$

where $c = c(x, \tau) \equiv \tau\mu\lambda(x)$ and \mathbf{V}_{em} is given by (3.47).

Proof. We prepare two elementary estimates. Since \mathbf{V}_e^0 satisfies (1.14), we have

$$\frac{1}{\mu\epsilon} \nabla \times \nabla \times \mathbf{V}_e^0 + \tau^2 \mathbf{V}_e^0 = \mathbf{0} \text{ in } D.$$

Thus (3.36) gives

$$\|\nabla \times \nabla \times \mathbf{V}_e^0\|_{L^2(D)} = O(\tau^{-1/2}).$$

By the trace theorem [5] this together with (3.37) yields

$$\|(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}\|_{L^2(\partial D)} = O(\tau^{-1/2}). \quad (4.3)$$

Next from equation (3.38) and (3.39) we have

$$\|\nabla \times \nabla \times (\mathbf{V}_e - \mathbf{V}_e^0)\|_{L^2(\mathbf{R}^3)} = O(\tau e^{-\tau T}).$$

Again by the trace theorem [5] this together with (3.40) gives

$$\|(\nabla \times (\mathbf{V}_e - \mathbf{V}_e^0)) \times \boldsymbol{\nu}\|_{L^2(\partial D)} = O(\tau e^{-\tau T}). \quad (4.4)$$

Moreover, we add also

$$\|\mathbf{V}_e^0 \times \boldsymbol{\nu}\|_{L^2(\partial D)} = O(\tau^{-3/2}) \quad (4.5)$$

and

$$\|\boldsymbol{\nu} \times (\mathbf{V}_e - \mathbf{V}_e^0)\|_{L^2(\partial D)} = O(e^{-\tau T}), \quad (4.6)$$

which are derived from the trace theorem [5] together with (3.36) and (3.37); (3.39) and (3.40), respectively.

Then, from (4.3) and (4.4) we have, in $L^2(\partial D)$

$$\begin{aligned} & \mathbf{V}_m \times \boldsymbol{\nu} \\ &= -\frac{1}{\tau\mu}(\nabla \times \mathbf{V}_e) \times \boldsymbol{\nu} + \frac{e^{-\tau T}}{\tau} \mathbf{H}_0 \times \boldsymbol{\nu} \\ &= -\frac{1}{\tau\mu}(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} - \frac{1}{\tau\mu}\{\nabla \times (\mathbf{V}_e - \mathbf{V}_e^0)\} \times \boldsymbol{\nu} + O(\tau^{-1}e^{-\tau T}) \\ &= -\frac{1}{\tau\mu}(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} + O(e^{-\tau T}). \end{aligned}$$

Thus we have

$$\begin{aligned} \int_{\partial D} \frac{\tau\mu}{\lambda} |\mathbf{V}_m \times \boldsymbol{\nu}|^2 dS &= \int_{\partial D} \frac{\tau\mu}{\lambda} \frac{1}{\tau^2\mu^2} |(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}|^2 dS \\ &\quad + O(\tau^{-1/2}e^{-\tau T}) + O(e^{-2\tau T}) \\ &= \int_{\partial D} \frac{1}{c} |(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}|^2 dS + O(\tau^{-1/2}e^{-\tau T}). \end{aligned} \quad (4.7)$$

Second write

$$\begin{aligned}
& (\boldsymbol{\nu} \times \mathbf{V}_e) \cdot \nabla \times \mathbf{V}_e \\
&= (\boldsymbol{\nu} \times \mathbf{V}_e) \cdot \boldsymbol{\nu} \times \{(\nabla \times \mathbf{V}_e) \times \boldsymbol{\nu}\} \\
&= (\boldsymbol{\nu} \times \mathbf{V}_e^0) \cdot \boldsymbol{\nu} \times \{(\nabla \times \mathbf{V}_e) \times \boldsymbol{\nu}\} + \{\boldsymbol{\nu} \times (\mathbf{V}_e - \mathbf{V}_e^0)\} \cdot \boldsymbol{\nu} \times \{(\nabla \times \mathbf{V}_e) \times \boldsymbol{\nu}\} \\
&= (\boldsymbol{\nu} \times \mathbf{V}_e^0) \cdot \boldsymbol{\nu} \times \{(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}\} + (\boldsymbol{\nu} \times \mathbf{V}_e^0) \cdot \boldsymbol{\nu} \times \{(\nabla \times (\mathbf{V}_e - \mathbf{V}_e^0)) \times \boldsymbol{\nu}\} \\
&\quad + \{\boldsymbol{\nu} \times (\mathbf{V}_e - \mathbf{V}_e^0)\} \cdot \boldsymbol{\nu} \times \{(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}\} \\
&\quad + \{\boldsymbol{\nu} \times (\mathbf{V}_e - \mathbf{V}_e^0)\} \cdot \boldsymbol{\nu} \times \{(\nabla \times (\mathbf{V}_e - \mathbf{V}_e^0)) \times \boldsymbol{\nu}\}.
\end{aligned}$$

Then using (4.3), (4.4), (4.5) and (4.6), we have

$$\begin{aligned}
& \int_{\partial D} (\boldsymbol{\nu} \times \mathbf{V}_e) \cdot \nabla \times \mathbf{V}_e dS \\
&= \int_{\partial D} (\boldsymbol{\nu} \times \mathbf{V}_e^0) \cdot \boldsymbol{\nu} \times \{(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}\} dS \\
&\quad + O(\tau^{-3/2} \tau e^{-\tau T}) + O(e^{-\tau T} \tau^{-1/2}) + O(e^{-\tau T} \tau e^{-\tau T}) = O(\tau^{-1/2} e^{-\tau T}).
\end{aligned} \tag{4.8}$$

Since

$$(\boldsymbol{\nu} \times \mathbf{V}_e^0) \cdot \boldsymbol{\nu} \times \{(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}\} = \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) \cdot \{(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}\}$$

and

$$\tilde{J}_e(\tau) = \frac{1}{\mu\epsilon} \left(\int_{\partial D} (\boldsymbol{\nu} \times \mathbf{V}_e) \cdot \nabla \times \mathbf{V}_e - \frac{\tau\mu}{\lambda} |\mathbf{V}_m \times \boldsymbol{\nu}|^2 \right) dS,$$

from (4.7) and (4.8) we obtain

$$\begin{aligned}
\tilde{J}_e(\tau) &= \frac{1}{\mu\epsilon} \int_{\partial D} \left\{ \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) \cdot \{(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}\} - \frac{1}{c} |(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}|^2 \right\} dS \\
&\quad + O(\tau^{-1/2} e^{-\tau T}).
\end{aligned}$$

This is nothing but (4.1).

Next, in $L^2(\partial D)$ from (4.3), (4.4) and (4.6) we have

$$\begin{aligned}
& \mathbf{V}_{em} \\
&= \boldsymbol{\nu} \times (\mathbf{V}_e \times \boldsymbol{\nu}) + \frac{1}{\lambda} \mathbf{V}_m \times \boldsymbol{\nu} \\
&= \boldsymbol{\nu} \times (\mathbf{V}_e \times \boldsymbol{\nu}) + \frac{1}{\lambda} \left\{ -\frac{1}{\tau\mu} (\nabla \times \mathbf{V}_e) \times \boldsymbol{\nu} - \frac{e^{-\tau T}}{\tau} \mathbf{H}_0 \right\} \times \boldsymbol{\nu} \\
&= \boldsymbol{\nu} \times (\mathbf{V}_e \times \boldsymbol{\nu}) - \frac{1}{c} (\nabla \times \mathbf{V}_e) \times \boldsymbol{\nu} - \frac{e^{-\tau T}}{\lambda\tau} \mathbf{H}_0 \times \boldsymbol{\nu} \\
&= \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) + \boldsymbol{\nu} \times \{(\mathbf{V}_e - \mathbf{V}_e^0) \times \boldsymbol{\nu}\} \\
&\quad - \frac{1}{c} (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} - \frac{1}{c} \{ \nabla \times (\mathbf{V}_e - \mathbf{V}_e^0) \} \times \boldsymbol{\nu} + O(\tau^{-1} e^{-\tau T}) \\
&= \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) + O(e^{-\tau T}) - \frac{1}{c} (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} + O(e^{-\tau T}) + O(\tau^{-1} e^{-\tau T}) \\
&= \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) - \frac{1}{c} (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} + O(e^{-\tau T}).
\end{aligned}$$

Then from (4.3) and (4.5) we obtain

$$\begin{aligned}
& \frac{1}{\mu\epsilon} \int_{\partial D} c |\mathbf{V}_{em}|^2 dS \\
&= \frac{1}{\mu\epsilon} \int_{\partial D} c \left| \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) - \frac{1}{c} (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} \right|^2 dS \\
&\quad + O(\tau\tau^{-3/2} e^{-\tau T}) + O(\tau e^{-2\tau T}) \\
&= \frac{1}{\mu\epsilon} \int_{\partial D} \frac{1}{c} \left| c \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) - (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} \right|^2 dS + O(\tau^{-1/2} e^{-\tau T}).
\end{aligned} \tag{4.9}$$

Now a combination of (4.1) and (4.9) yields (4.2).

4.2 Preliminary computation for the proof of (i) and (ii)

4.2.1 Computation of $\boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu})$

By (18) in [9] we know that \mathbf{V}_e^0 is smooth outside B and has the form

$$\mathbf{V}_e^0 = K(\tau) \tilde{f}(\tau) v \mathbf{M} \mathbf{a} \quad \text{in } \mathbf{R}^3 \setminus \overline{B}, \tag{4.10}$$

where

$$\begin{cases} v = v(x) = \frac{e^{-\tau\sqrt{\mu\epsilon}|x-p|}}{|x-p|}, \\ K(\tau) = \frac{\mu\tau\varphi(\tau\sqrt{\mu\epsilon})}{(\tau\sqrt{\mu\epsilon})^3}, \\ \varphi(\xi) = \xi \cosh \xi - \sinh \xi \end{cases}$$

and

$$\begin{cases} \mathbf{M} = \mathbf{M}(x; \tau) = A\mathbf{I}_3 - B \frac{x-p}{|x-p|} \otimes \frac{x-p}{|x-p|}, \\ A = A(x, \tau) = 1 + \frac{1}{\tau\sqrt{\mu\epsilon}} \left(\frac{1}{|x-p|} + \frac{1}{\tau\sqrt{\mu\epsilon}|x-p|^2} \right), \\ B = B(x, \tau) = 1 + \frac{3}{\tau\sqrt{\mu\epsilon}} \left(\frac{1}{|x-p|} + \frac{1}{\tau\sqrt{\mu\epsilon}|x-p|^2} \right). \end{cases}$$

The expression (4.10) is a simple application of the mean value theorem [3] for the modified Helmholtz equation and a special form of the fundamental solution of equation (1.14).

From (4.10) we have

$$\boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) = K(\tau)\tilde{f}(\tau)v \boldsymbol{\nu} \times \{(\mathbf{M}\mathbf{a}) \times \boldsymbol{\nu}\}.$$

4.2.2 Computation of $(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}$

By (23) in [9], from (4.10) we have already derived the expression of $\nabla \times \mathbf{V}_e^0$ outside B :

$$\nabla \times \mathbf{V}_e^0 = -\tau\sqrt{\mu\epsilon} K(\tau)\tilde{f}(\tau)v \left(1 + \frac{1}{\tau\sqrt{\mu\epsilon}|x-p|} \right) \frac{x-p}{|x-p|} \times \mathbf{a}.$$

Thus we obtain

$$(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} = -\tau\sqrt{\mu\epsilon} K(\tau)\tilde{f}(\tau)v \left(1 + \frac{1}{\tau\sqrt{\mu\epsilon}|x-p|} \right) \left(\frac{x-p}{|x-p|} \times \mathbf{a} \right) \times \boldsymbol{\nu}.$$

4.2.3 Computation of $\boldsymbol{\nu} \times \{(\mathbf{M}\mathbf{a}) \times \boldsymbol{\nu}\}$

We have

$$\mathbf{M}\mathbf{a} = A\mathbf{a} - B \left(\mathbf{a} \cdot \frac{x-p}{|x-p|} \right) \frac{x-p}{|x-p|}$$

and

$$(\mathbf{M}\mathbf{a}) \cdot \boldsymbol{\nu} = A\mathbf{a} \cdot \boldsymbol{\nu} - B \left(\mathbf{a} \cdot \frac{x-p}{|x-p|} \right) \left(\boldsymbol{\nu} \cdot \frac{x-p}{|x-p|} \right).$$

Combining these with expression

$$\boldsymbol{\nu} \times \{(\mathbf{M}\mathbf{a}) \times \boldsymbol{\nu}\} = \mathbf{M}\mathbf{a} - \{(\mathbf{M}\mathbf{a}) \cdot \boldsymbol{\nu}\}\boldsymbol{\nu},$$

we obtain

$$\boldsymbol{\nu} \times \{(\mathbf{M}\mathbf{a}) \times \boldsymbol{\nu}\} = A\boldsymbol{\nu} \times (\mathbf{a} \times \boldsymbol{\nu}) - B \left(\mathbf{a} \cdot \frac{x-p}{|x-p|} \right) \boldsymbol{\nu} \times \left(\frac{x-p}{|x-p|} \times \boldsymbol{\nu} \right).$$

4.2.4 Computation of $\left(\frac{x-p}{|x-p|} \times \mathbf{a}\right) \times \boldsymbol{\nu}$

From the identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$, we obtain

$$\begin{aligned}
& \left(\frac{x-p}{|x-p|} \times \mathbf{a}\right) \times \boldsymbol{\nu} \\
&= -\boldsymbol{\nu} \times \left(\frac{x-p}{|x-p|} \times \mathbf{a}\right) \\
&= \boldsymbol{\nu} \times \left(\mathbf{a} \times \frac{x-p}{|x-p|}\right) \\
&= \left(\boldsymbol{\nu} \cdot \frac{x-p}{|x-p|}\right) \mathbf{a} - (\boldsymbol{\nu} \cdot \mathbf{a}) \frac{x-p}{|x-p|} \\
&= \left(\boldsymbol{\nu} \cdot \frac{x-p}{|x-p|}\right) \boldsymbol{\nu} \times (\mathbf{a} \times \boldsymbol{\nu}) - (\boldsymbol{\nu} \cdot \mathbf{a}) \boldsymbol{\nu} \times \left(\frac{x-p}{|x-p|} \times \boldsymbol{\nu}\right)
\end{aligned}$$

4.2.5 Computation of $c \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) - (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}$

We have

$$\begin{aligned}
& c \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) - (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} \\
&= K(\tau) \tilde{f}(\tau) v \\
&\quad \times \left\{ c \boldsymbol{\nu} \times \{(\mathbf{M} \mathbf{a}) \times \boldsymbol{\nu}\} + \tau \sqrt{\mu \epsilon} \left(1 + \frac{1}{\tau \sqrt{\mu \epsilon} |x-p|}\right) \left(\frac{x-p}{|x-p|} \times \mathbf{a}\right) \times \boldsymbol{\nu} \right\} \\
&= K(\tau) \tilde{f}(\tau) \tau \mu \\
&\quad \times \left\{ \lambda(x) \boldsymbol{\nu} \times \{(\mathbf{M} \mathbf{a}) \times \boldsymbol{\nu}\} + \sqrt{\frac{\epsilon}{\mu}} \left(1 + \frac{1}{\tau \sqrt{\mu \epsilon} |x-p|}\right) \left(\frac{x-p}{|x-p|} \times \mathbf{a}\right) \times \boldsymbol{\nu} \right\} \\
&= K(\tau) \tilde{f}(\tau) \tau \mu v (P \mathbf{X} - Q \mathbf{Y}),
\end{aligned}$$

where

$$\begin{cases} P = \lambda A + \sqrt{\frac{\epsilon}{\mu}} \left(1 + \frac{1}{\tau \sqrt{\mu \epsilon} |x-p|}\right) \left(\boldsymbol{\nu} \cdot \frac{x-p}{|x-p|}\right), \\ Q = \lambda B \left(\mathbf{a} \cdot \frac{x-p}{|x-p|}\right) + \sqrt{\frac{\epsilon}{\mu}} \left(1 + \frac{1}{\tau \sqrt{\mu \epsilon} |x-p|}\right) (\boldsymbol{\nu} \cdot \mathbf{a}) \end{cases}$$

and

$$\mathbf{X} = \boldsymbol{\nu} \times (\mathbf{a} \times \boldsymbol{\nu}), \quad \mathbf{Y} = \boldsymbol{\nu} \times \left(\frac{x-p}{|x-p|} \times \boldsymbol{\nu}\right).$$

4.2.6 Computation of $\{c\boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) - (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}\} \cdot (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}$

We have

$$(\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} = -\tau \sqrt{\mu \epsilon} K(\tau) \tilde{f}(\tau) v(x) \left(1 + \frac{1}{\tau \sqrt{\mu \epsilon} |x - p|}\right) \left\{ \left(\boldsymbol{\nu} \cdot \frac{x - p}{|x - p|} \right) \mathbf{X} - (\boldsymbol{\nu} \cdot \mathbf{a}) \mathbf{Y} \right\}.$$

Thus we obtain

$$\begin{aligned} & \{c\boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) - (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}\} \cdot (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu} \\ &= -\tau^2 \mu \sqrt{\mu \epsilon} K(\tau)^2 (\tilde{f}(\tau))^2 v(x)^2 \left(1 + \frac{1}{\tau \sqrt{\mu \epsilon} |x - p|}\right) \\ & \quad \times (P \mathbf{X} - Q \mathbf{Y}) \cdot \left\{ \left(\boldsymbol{\nu} \cdot \frac{x - p}{|x - p|} \right) \mathbf{X} - (\boldsymbol{\nu} \cdot \mathbf{a}) \mathbf{Y} \right\}. \end{aligned} \quad (4.11)$$

4.2.7 Computation of $\{c\boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) - (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}\} \cdot \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu})$

We have

$$\boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) = K(\tau) \tilde{f}(\tau) v(x) \left\{ A \mathbf{X} - B \left(\mathbf{a} \cdot \frac{x - p}{|x - p|} \right) \mathbf{Y} \right\}.$$

Therefore we obtain

$$\begin{aligned} & \{c\boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) - (\nabla \times \mathbf{V}_e^0) \times \boldsymbol{\nu}\} \cdot \boldsymbol{\nu} \times (\mathbf{V}_e^0 \times \boldsymbol{\nu}) \\ &= \tau \mu K(\tau)^2 (\tilde{f}(\tau))^2 v(x)^2 (P \mathbf{X} - Q \mathbf{Y}) \cdot \left\{ A \mathbf{X} - B \left(\mathbf{a} \cdot \frac{x - p}{|x - p|} \right) \mathbf{Y} \right\}. \end{aligned} \quad (4.12)$$

4.3 Proof of (i)

In this and next subsections, to emphasize the dependence on space variable x we write $P = P_x$, $Q = Q_x$, $\mathbf{X} = \mathbf{X}_x$, $\mathbf{Y} = \mathbf{Y}_x$ and $\boldsymbol{\nu} = \boldsymbol{\nu}_x$.

First we prepare an elementary lemma which can be proved by using a contradiction argument.

Lemma 4.2 *Given $\eta > 0$ there exists a positive number δ such that, for all $x \in \partial D \cap B_{d_{\partial D}(p)+\delta}(p)$*

$$\boldsymbol{\nu}_x \cdot \frac{p - x}{|x - p|} > 1 - \eta.$$

Since

$$|\mathbf{Y}_x|^2 = 1 - \left(\frac{p - x}{|x - p|} \cdot \boldsymbol{\nu}_x \right)^2,$$

it follows from Lemma 4.2 that, for all $x \in \partial D \cap B_{d_{\partial D}(p)+\delta}(p)$

$$|\mathbf{Y}_x| \leq \sqrt{2\eta}. \quad (4.13)$$

Since \mathbf{a}_1 and \mathbf{a}_2 are linearly independent, it is easy to see that, for all $x \in \partial D$ we have $\sum_{j=1}^2 |\boldsymbol{\nu}_x \times (\mathbf{a}_j \times \boldsymbol{\nu}_x)|^2 > 0$. Since ∂D is compact, this yields that $\min_{x \in \partial D} \sum_{j=1}^2 |\boldsymbol{\nu}_x \times (\mathbf{a}_j \times \boldsymbol{\nu}_x)|^2 > 0$. Choose a sufficiently small $\eta > 0$ in such a way that

$$\eta < \min_{x \in \partial D} \sum_{j=1}^2 |\boldsymbol{\nu}_x \times (\mathbf{a}_j \times \boldsymbol{\nu}_x)|^2.$$

We have

$$\begin{aligned} & -(P_x \mathbf{X}_x - Q_x \mathbf{Y}_x) \cdot \left\{ \left(\boldsymbol{\nu}_x \cdot \frac{x-p}{|x-p|} \right) \mathbf{X}_x - (\boldsymbol{\nu}_x \cdot \mathbf{a}_j) \mathbf{Y}_x \right\} \\ &= P_x \left(\boldsymbol{\nu}_x \cdot \frac{p-x}{|x-p|} \right) |\mathbf{X}_x|^2 \\ & \quad - Q_x (\boldsymbol{\nu}_x \cdot \mathbf{a}_j) |\mathbf{Y}_x|^2 + \left\{ P_x (\boldsymbol{\nu}_x \cdot \mathbf{a}_j) - Q_x \left(\boldsymbol{\nu}_x \cdot \frac{p-x}{|x-p|} \right) \right\} \mathbf{Y}_x \cdot \mathbf{X}_x. \end{aligned} \tag{4.14}$$

Write

$$\begin{aligned} P_x &= \lambda(x) \left\{ 1 + \frac{1}{\tau \sqrt{\mu \epsilon}} \left(\frac{1}{|x-p|} + \frac{1}{\tau \sqrt{\mu \epsilon} |x-p|^2} \right) \right\} \\ & \quad - \sqrt{\frac{\epsilon}{\mu}} \left(1 + \frac{1}{\tau \sqrt{\mu \epsilon} |x-p|} \right) \boldsymbol{\nu}_x \cdot \frac{p-x}{|x-p|} \\ &= \left(1 + \frac{1}{\tau \sqrt{\mu \epsilon} |x-p|} \right) \left(\lambda(x) - \sqrt{\frac{\epsilon}{\mu}} \boldsymbol{\nu}_x \cdot \frac{p-x}{|x-p|} \right) + \frac{\lambda(x)}{(\tau \sqrt{\mu \epsilon})^2 |x-p|^2}. \end{aligned} \tag{4.15}$$

Since we have, for all $x \in \partial D$

$$\begin{aligned} \lambda(x) - \sqrt{\frac{\epsilon}{\mu}} \boldsymbol{\nu}_x \cdot \frac{p-x}{|x-p|} &= \left(\lambda(x) - \sqrt{\frac{\epsilon}{\mu}} \right) + \sqrt{\frac{\epsilon}{\mu}} \left(1 - \boldsymbol{\nu}_x \cdot \frac{p-x}{|x-p|} \right) \\ &\geq \lambda(x) - \sqrt{\frac{\epsilon}{\mu}} \geq C, \end{aligned}$$

it follows from these and Lemma 4.2 for $\eta < 1/2$ we obtain, for all $\tau > 0$ and $x \in \partial D \cap B_{d_{\partial D}(p)+\delta}(p)$

$$P_x \left(\boldsymbol{\nu}_x \cdot \frac{p-x}{|x-p|} \right) \geq \frac{C}{2}. \tag{4.16}$$

Using (4.13) and simply estimating from above, we have

$$\begin{aligned} & \left| -Q_x (\boldsymbol{\nu}_x \cdot \mathbf{a}_j) |\mathbf{Y}_x|^2 + \left\{ P_x (\boldsymbol{\nu}_x \cdot \mathbf{a}_j) - Q_x \left(\boldsymbol{\nu}_x \cdot \frac{p-x}{|x-p|} \right) \right\} \mathbf{Y}_x \cdot \mathbf{X}_x \right| \\ &\leq C'(\eta + \sqrt{\eta} |\mathbf{X}_x|). \end{aligned}$$

Applying this together with (4.16) to (4.14), we have, for all $\kappa > 0$

$$\begin{aligned}
& - (P_x \mathbf{X}_x - Q_x \mathbf{Y}_x) \cdot \left\{ \left(\boldsymbol{\nu}_x \cdot \frac{x-p}{|x-p|} \right) \mathbf{X}_x - (\boldsymbol{\nu}_x \cdot \mathbf{a}_j) \mathbf{Y}_x \right\} \\
& \geq \frac{C}{2} |\mathbf{X}_x|^2 - C'(\eta + \sqrt{\eta} |\mathbf{X}_x|) \\
& \geq \left(\frac{C}{2} - \frac{C'\kappa}{2} \right) |\mathbf{X}_x|^2 - C' \left(\eta + \frac{\kappa^{-1}\eta}{2} \right).
\end{aligned}$$

Thus, letting $\kappa = C/(2C')$, we obtain, for all $x \in \partial D \cap B_{d_{\partial D}(p)+\delta}(p)$ and $\tau > 0$

$$- (P_x \mathbf{X}_x - Q_x \mathbf{Y}_x) \cdot \left\{ \left(\boldsymbol{\nu}_x \cdot \frac{x-p}{|x-p|} \right) \mathbf{X}_x - (\boldsymbol{\nu}_x \cdot \mathbf{a}_j) \mathbf{Y}_x \right\} \geq \frac{C}{4} |\mathbf{X}_x|^2 - C_2 \eta,$$

where $C_2 = C'(C + C')/C$.

Now it follows from this and (4.11) that, for all $x \in \partial D \cap B_{d_{\partial D}(p)+\delta}(p)$

$$\begin{aligned}
& \sum_{j=1}^2 \frac{1}{c} \left\{ c \boldsymbol{\nu}_x \times (\mathbf{V}_{e,j}^0(x) \times \boldsymbol{\nu}_x) - (\nabla \times \mathbf{V}_{e,j}^0(x)) \times \boldsymbol{\nu}_x \right\} \cdot (\nabla \times \mathbf{V}_{e,j}^0(x)) \times \boldsymbol{\nu}_x \\
& \geq (\lambda(x))^{-1} \sqrt{\mu\epsilon} \tau K(\tau)^2 (\tilde{f}(\tau))^2 v(x)^2 \left(\frac{C}{4} \sum_{j=1}^2 |\boldsymbol{\nu}_x \times (\mathbf{a}_j \times \boldsymbol{\nu}_x)|^2 - 2C_2 \eta \right).
\end{aligned} \tag{4.17}$$

Now re-choosing η in such a way that

$$\eta < \frac{C}{16C_2} \min_{x \in \partial D} \sum_{j=1}^2 |\boldsymbol{\nu}_x \times (\mathbf{a}_j \times \boldsymbol{\nu}_x)|^2,$$

we have

$$\begin{aligned}
& (\lambda(x))^{-1} \sqrt{\mu\epsilon} \tau K(\tau)^2 (\tilde{f}(\tau))^2 v(x)^2 \left(\frac{C}{4} \sum_{j=1}^2 |\boldsymbol{\nu}_x \times (\mathbf{a}_j \times \boldsymbol{\nu}_x)|^2 - 2C_4 \eta \right) \\
& \geq (\|\lambda\|_{L^\infty((\partial D)_\delta(p))})^{-1} \sqrt{\mu\epsilon} \tau K(\tau)^2 (\tilde{f}(\tau))^2 v(x)^2 \frac{C}{8} \min_{x \in \partial D} \sum_{j=1}^2 |\boldsymbol{\nu}_x \times (\mathbf{a}_j \times \boldsymbol{\nu}_x)|^2.
\end{aligned}$$

Therefore this together with (4.17) yields

$$\begin{aligned}
& \sum_{j=1}^2 \frac{1}{\mu\epsilon} \int_{\partial D \cap B_{d_{\partial D}(p)+\delta}(p)} \frac{1}{c} \left\{ c \boldsymbol{\nu} \times (\mathbf{V}_{e,j}^0 \times \boldsymbol{\nu}) - (\nabla \times \mathbf{V}_{e,j}^0) \times \boldsymbol{\nu} \right\} \cdot (\nabla \times \mathbf{V}_{e,j}^0) \times \boldsymbol{\nu} dS \\
& \geq C_3 (\|\lambda\|_{L^\infty(\partial D)} \sqrt{\mu\epsilon})^{-1} \tau K(\tau)^2 (\tilde{f}(\tau))^2 \int_{\partial D \cap B_{d_{\partial D}(p)+\delta}(p)} v(x)^2 dS,
\end{aligned} \tag{4.18}$$

where

$$C_3 = \frac{C}{8} \min_{x \in \partial D} \sum_{j=1}^2 |\boldsymbol{\nu}_x \times (\mathbf{a}_j \times \boldsymbol{\nu}_x)|^2.$$

Using the same argument done in the proof of Lemma 2.2 in [7], we know that there exists a positive constant C'' such that, for all $\tau \gg 1$

$$\int_{\partial D \cap B_{d_{\partial D}(p)+\delta}(p)} v(x)^2 dS \geq C'' \tau^{-4} e^{-2\tau\sqrt{\mu\epsilon}d_{\partial D}(p)}.$$

And it is easy to see that, as $\tau \rightarrow \infty$

$$K(\tau) \sim \tau^{-1} \frac{\eta e^{\tau\eta\sqrt{\mu\epsilon}}}{2\epsilon}. \quad (4.19)$$

Therefore we have

$$\tau K(\tau)^2 \int_{\partial D \cap B_{d_{\partial D}(p)+\delta}(p)} v(x)^2 dS \geq C''' \tau^{-5} e^{-2\tau\sqrt{\mu\epsilon}\text{dist}(D,B)}, \quad (4.20)$$

where C''' is a positive constant and $\tau \gg 1$. Since \mathbf{V}_e^0 together with its derivatives on $\partial D \setminus B_{d_{\partial D}(p)+\delta}(p)$ is decaying as $e^{-\tau\sqrt{\mu\epsilon}\text{dist}(D,B)} e^{-\tau\sqrt{\mu\epsilon}\delta}$ (see (4.10) and note that $d_{\partial D}(p) - \eta = \text{dist}(D, B)$), (4.1), (4.18), (4.20) and (1.11) yields (i) with $\rho = 2\gamma + 5$.

4.4 Proof of (ii)

Write

$$\begin{aligned} & (P_x \mathbf{X}_x - Q_x \mathbf{Y}_x) \cdot \left\{ A \mathbf{X}_x - B \left(\mathbf{a} \cdot \frac{x-p}{|x-p|} \right) \mathbf{Y}_x \right\} \\ &= P_x A |\mathbf{X}|^2 \\ & - \left\{ Q_x A + P_x B \left(\mathbf{a} \cdot \frac{x-p}{|x-p|} \right) \right\} \mathbf{X}_x \cdot \mathbf{Y}_x + Q_x B \left(\mathbf{a} \cdot \frac{x-p}{|x-p|} \right) |\mathbf{Y}_x|^2. \end{aligned} \quad (4.21)$$

It follows from Lemma 4.2 that, for all $x \in \partial D \cap B_{d_{\partial D}(p)+\delta}(p)$

$$\begin{aligned} \lambda(x) - \sqrt{\frac{\epsilon}{\mu}} \boldsymbol{\nu}_x \cdot \frac{p-x}{|x-p|} &= \left(\lambda(x) - \sqrt{\frac{\epsilon}{\mu}} \right) + \sqrt{\frac{\epsilon}{\mu}} \left(1 - \boldsymbol{\nu}_x \cdot \frac{p-x}{|x-p|} \right) \\ &\leq -C + \sqrt{\frac{\epsilon}{\mu}} \eta. \end{aligned}$$

Thus, choosing η in such a way that

$$\eta < \frac{C}{2} \sqrt{\frac{\mu}{\epsilon}},$$

we have

$$\lambda(x) - \sqrt{\frac{\epsilon}{\mu}} \boldsymbol{\nu}_x \cdot \frac{p-x}{|x-p|} = \left(\lambda(x) - \sqrt{\frac{\epsilon}{\mu}} \right) + \sqrt{\frac{\epsilon}{\mu}} \left(1 - \boldsymbol{\nu}_x \cdot \frac{p-x}{|x-p|} \right) \leq -\frac{C}{2}.$$

Therefore from (4.15) we obtain

$$P_x \leq -\frac{C}{2} + \frac{\|\lambda\|_{L^\infty(\partial D)}}{(\tau\sqrt{\mu\epsilon})^2(d_{\partial D}(p))^2}.$$

Thus, choosing $\tau_0 > 0$ in such a way that

$$\frac{\|\lambda\|_{L^\infty(\partial D)}}{(\tau_0\sqrt{\mu\epsilon})^2(d_{\partial D}(p))^2} = \frac{C}{4},$$

we have, for all $\tau \geq \tau_0$ and all $x \in \partial D \cap B_{d_{\partial D}(p)+\delta}(p)$

$$P_x \leq -\frac{C}{4}.$$

Since $A \geq 1$, this yields

$$P_x A \leq -\frac{C}{4}. \quad (4.22)$$

It is easy to see that we have

$$\begin{aligned} & \left| -\left\{ Q_x A + P_x B \left(\mathbf{a} \cdot \frac{x-p}{|x-p|} \right) \right\} \mathbf{X}_x \cdot \mathbf{Y}_x + Q_x B \left(\mathbf{a} \cdot \frac{x-p}{|x-p|} \right) |\mathbf{Y}_x|^2 \right| \\ & \leq C_4(\eta + \sqrt{\eta}|\mathbf{X}_x|). \end{aligned}$$

Applying this together with (4.22) to (4.21), we obtain

$$\begin{aligned} & (P_x \mathbf{X}_x - Q_x \mathbf{Y}_x) \cdot \left\{ A \mathbf{X}_x - B \left(\mathbf{a} \cdot \frac{x-p}{|x-p|} \right) \mathbf{Y}_x \right\} \\ & \leq -\frac{C}{4}|\mathbf{X}_x|^2 + C_4(\eta + \sqrt{\eta}|\mathbf{X}_x|) \\ & \leq -\left(\frac{C}{4} - \frac{C_4\kappa}{2} \right) |\mathbf{X}_x|^2 + C_4\eta \left(1 + \frac{1}{2\kappa} \right), \end{aligned}$$

where κ is an arbitrary positive number. Thus letting $\kappa = C/(4C_4)$, we obtain

$$(P_x \mathbf{X}_x - Q_x \mathbf{Y}_x) \cdot \left\{ A \mathbf{X}_x - B \left(\mathbf{a} \cdot \frac{x-p}{|x-p|} \right) \mathbf{Y}_x \right\} \leq -\left(\frac{C}{8} |\mathbf{X}_x|^2 - C_5\eta \right),$$

where $C_5 = C_4(C + 2C_4)/C$.

Therefore this together with (4.12) yields that, for all $x \in \partial D \cap B_{d_{\partial D}(p)+\delta}(p)$ and all $\tau \geq \tau_0$

$$\begin{aligned} & \sum_{j=1}^2 \left\{ c \boldsymbol{\nu}_x \times (\mathbf{V}_{e,j}^0(x) \times \boldsymbol{\nu}_x) - (\nabla \times \mathbf{V}_{e,j}^0(x)) \times \boldsymbol{\nu}_x \right\} \cdot \boldsymbol{\nu}_x \times (\mathbf{V}_{e,j}^0(x) \times \boldsymbol{\nu}_x) \\ & \leq -\mu\tau K(\tau)^2(\tilde{f}(\tau))^2 v(x)^2 \left(\frac{C}{8} \sum_{j=1}^2 |\boldsymbol{\nu}_x \times (\mathbf{a}_j \times \boldsymbol{\nu}_x)|^2 - 2C_5\eta \right). \end{aligned} \quad (4.23)$$

Now re-choosing η in such a way that

$$\eta < \frac{C}{32C_5} \min_{x \in \partial D} \sum_{j=1}^2 |\boldsymbol{\nu}_x \times (\mathbf{a}_j \times \boldsymbol{\nu}_x)|^2,$$

from (4.23) we obtain

$$\begin{aligned} & \sum_{j=1}^2 \frac{1}{\mu\epsilon} \int_{\partial D \cap B_{d_{\partial D}(p)+\delta}(p)} \left\{ c \boldsymbol{\nu} \times (\mathbf{V}_{e,j}^0 \times \boldsymbol{\nu}) - (\nabla \times \mathbf{V}_{e,j}^0) \times \boldsymbol{\nu} \right\} \cdot \boldsymbol{\nu} \times (\mathbf{V}_{e,j}^0 \times \boldsymbol{\nu}) dS \\ & \leq -C_6 \epsilon^{-1} \tau K(\tau)^2 (\tilde{f}(\tau))^2 \int_{\partial D \cap B_{d_{\partial D}(p)+\delta}(p)} v(x)^2 dS, \end{aligned}$$

where

$$C_6 = \frac{C}{16} \min_{x \in \partial D} \sum_{j=1}^2 |\boldsymbol{\nu}_x \times (\mathbf{a}_j \times \boldsymbol{\nu}_x)|^2.$$

Hereafter the procedure is completely same as that of the proof of (i).

4.5 Proof of Lemma 2.3

From (4.1), (4.11) and (4.19) we have, as $\tau \rightarrow \infty$

$$\tilde{J}_e(\tau) = O\left(\tau^{-1}(\tilde{f}(\tau))^2 e^{2\tau\sqrt{\mu\epsilon}} \int_{\partial D} v^2 dS\right) + O(\tau^{-1/2} e^{-\tau T})$$

and also (4.2), (4.11) and (4.19) give

$$\tilde{J}_e(\tau) + \frac{1}{\mu\epsilon} \int_{\partial D} c |\mathbf{V}_{em}|^2 dS = O\left(\tau^{-1}(\tilde{f}(\tau))^2 e^{2\tau\sqrt{\mu\epsilon}} \int_{\partial D} v^2 dS\right) + O(\tau^{-1/2} e^{-\tau T}).$$

Then, it follows from these, (2.1) and (2.2) that (2.5) is valid.

5 Conclusions and further problems

We have succeeded in extending the previous result in [7] for the scalar wave equation with the dissipative boundary condition to the Maxwell system with the Leontovich boundary condition. It can be also considered as an extension of Theorem 1.1 in [9] for the Maxwell system with the perfect conductivity condition to the Leontovich boundary condition. The main difference is to introduce an indicator function which employs two sets of electric fields corresponding to input sources oriented to two independent directions:

$$\tau \mapsto \sum_{j=1}^2 I_{\mathbf{f}_j}(\tau). \quad (5.1)$$

Technically the previous indicator function of Theorem 1.1 in [9] is based on the behaviour of a *volume integral* of \mathbf{V}_e^0 over the obstacle which is reduced to that of the fundamental solution of the modified Helmholtz equation. In contrast to the previous case, as a consequence of the presence of the impedance λ on the obstacle, the asymptotic

behaviour of each of two components of the new indicator function is governed by the behaviour of a *surface integral* of functions involving \mathbf{V}_e^0 together with derivatives. Since \mathbf{V}_e^0 has a *directivity*, it will be difficult to obtain the same result as Theorem 1.1 in [9] by using only a single component of the indicator function without assuming some restrictive condition for the source direction like (9) in [9].

Our finding is: to obtain the distance of a given point to an unknown obstacle together with a qualitative state of the surface of the obstacle *without* any restriction on the orientation of the source it is enough to make use of two sets of electric fields generated by two linearly independent input sources supported on a common open ball.

In future research [11] we will consider: extract information about the *curvatures* and impedance at the points on the surface of the obstacle nearest to the center of the support of input sources from the asymptotic behaviour of indicator function (5.1). This can be considered as an extension of Theorem 1.2 in [9]. And also it would be interested to consider the extension of the results in [8] to the Maxwell system. See [10] for a survey on other results using the enclosure method in the time domain together with other open problems.

Acknowledgment

The author was partially supported by Grant-in-Aid for Scientific Research (C)(No. 25400155) of Japan Society for the Promotion of Science.

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e-mail address

ikehata@amath.hiroshima-u.ac.jp